研究ノート

TRANSFORMATIONS OF HEEGAARD DIAGRAMS CORRESPONDING TO THOSE OF HEEGAARD CUT DIAGRAMS II[1]

Shunzi HORIGUCHI

1. Introduction

This paper is continued from the last number [**10**]. We use the same notations, definitions as in [**10**].

Theorem 3[2]**.** *Let* $U \cup V$ *be a Heegaard splitting of* M^3 [3]. *U, V is a Heegaard-handlebody, respectively. Let* $(U\,;\,m,\,l) \cup (V\,;\,l,\,m)$ *be Heegaard diagrams associated with* $U \cup V$. $\{m,\,l\}$ *is a meridian-longitude system of* $(U\,;\,m,\,l)$.

(7) *Let the following figure* $U6-A$ *be a part of* $(U\,;\,m,\,l)$. *The longitude* l_i *crosses the meridians* $\{m_1,\,\cdots,\,m_g\}$. *Let* $U6-B$ *be* $g-3$ *handles added to* $U6-A$. l_i *is decomposed into* $g-2$ *circles* $\{\tilde{l}_1,\,\cdots,\,\tilde{l}_{g-2}\}$ *and each circle goes around on three handles. Then there exists a transformation from* $U6-A$ *into* $U6-B$ *of* $(U\,;\,m,\,l)$.

(7′) *Let* $V6-A'$ *be a part of* $(V\,;\,l,\,m)$. $V6-A'$ *is the dual part of* $U6-A$, *i.e., each longitude* m_i $(i=1,\,\cdots,\,g)$ *goes around on the handle* h_i'. *We cut off* V *at the meridian disk* D_i' $(\partial D_i' = l_i)$ *and put* $g-2$ *handles over the two cutting places as shown in* $V6-B'$. *Then there exists a transformation from* $V6-A'$ *into* $V6-B'$ *of* $(V\,;\,l,\,m)$ *so that* m_i *goes around on the new handle.*

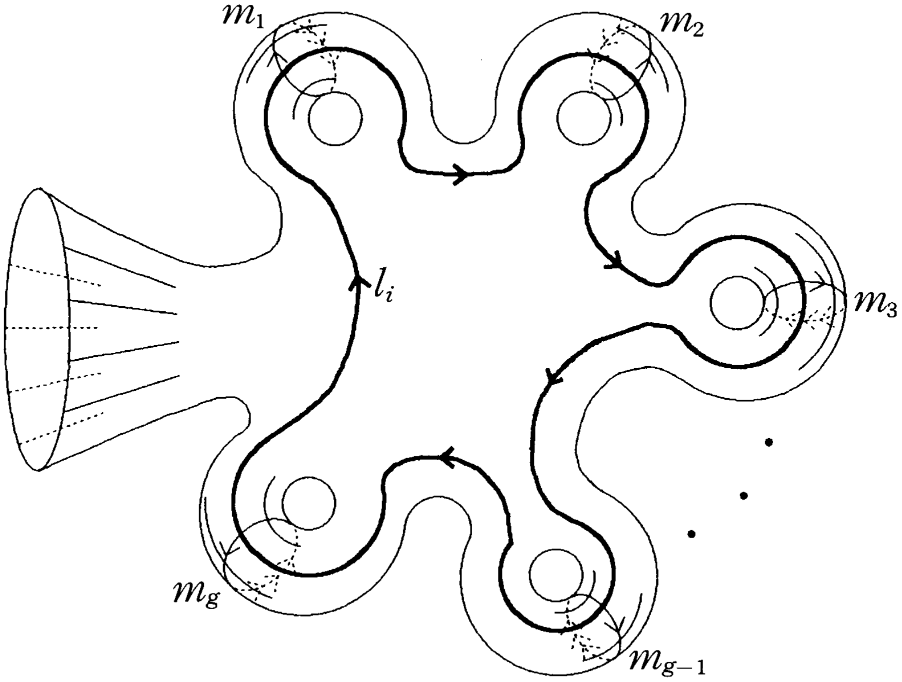
(8) *Let the following figure* $U7-A$ *be a part of* $(U\,;\,m,\,l)$. *The longitude* l_i *goes around on two handles and* l_j *goes around on another two handles. Let* $U7-B$ *be a handle added to* $U7-A$. *Then there exists a transformation from* $U7-A$ *into* $U7-B$ *of* $(U\,;\,m,\,l)$ *so that* l_i *and* l_j *go around on the handle.*

(8′) *Let* $V7-A'$ *be a part of* $(V\,;\,l,\,m)$. $V7-A'$ *is the dual part of* $U7-A$. *Let* $V7-B'$ *be a handle added to* $V7-A'$. *Then there exists a transformation from* $V7-A'$ *into* $V7-B'$ *of* $(V\,;\,l,\,m)$ *so that the longitude* $\tilde{m}$ *crosses the meridians* l_i, l_j *and* $\tilde{l}$.

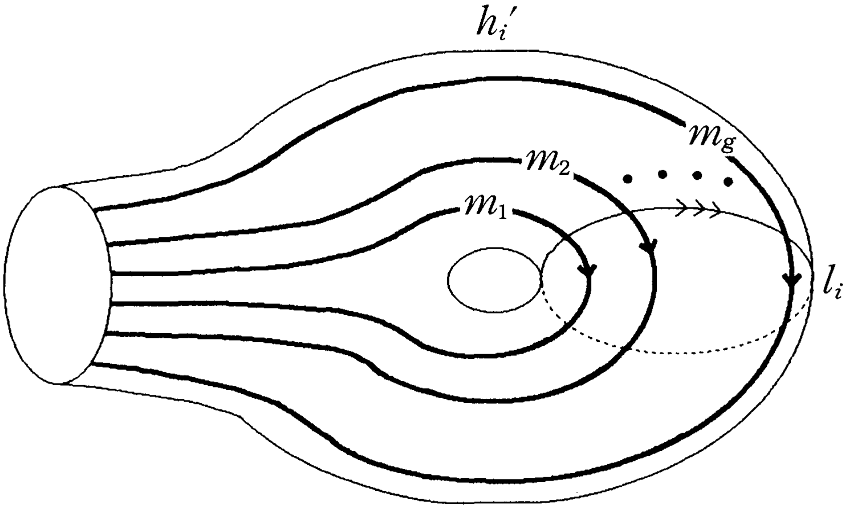
[1] Partially supported by the Mathematical Economics Study Group of Niigata Sangyou University.

[2] We use Theorem 3 as a serial notation.

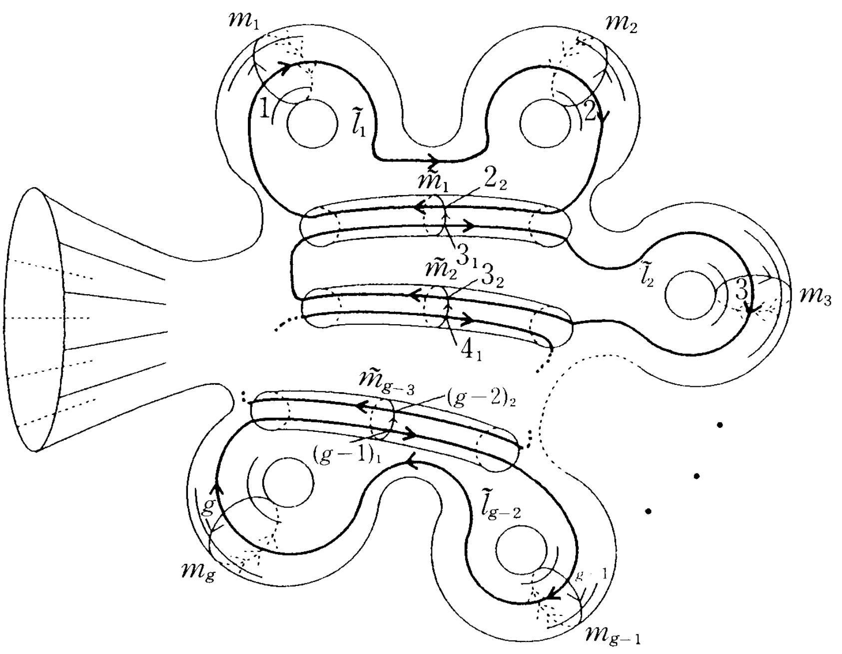
[3] M^3 denotes a connected orientable closed 3-manifold.



U6−A

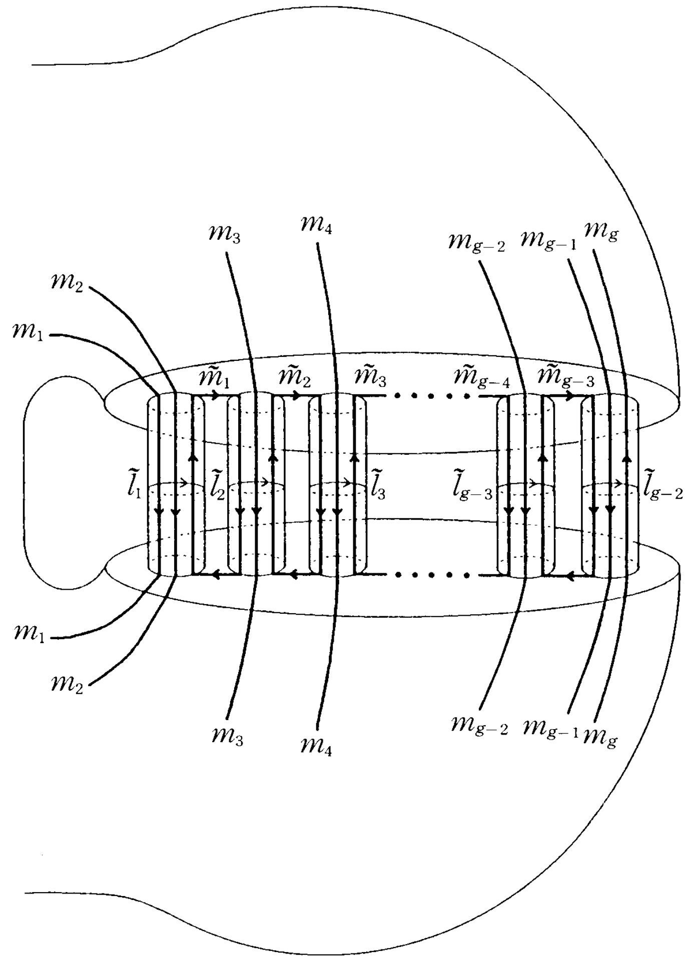


V6−A′



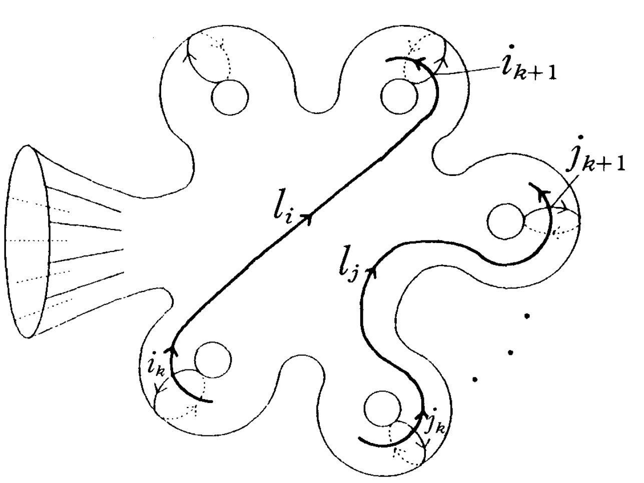
m_1
m_2
1
2
$\tilde{l}_1$
$\tilde{m}_1$
2_2
3_1
$\tilde{m}_2$
3_2
4_1
$\tilde{l}_2$
3
m_3
$\tilde{m}_{g-3}$
$(g-2)_2$
$(g-1)_1$
$\tilde{l}_{g-2}$
g
m_g
m_{g-1}

U6−B

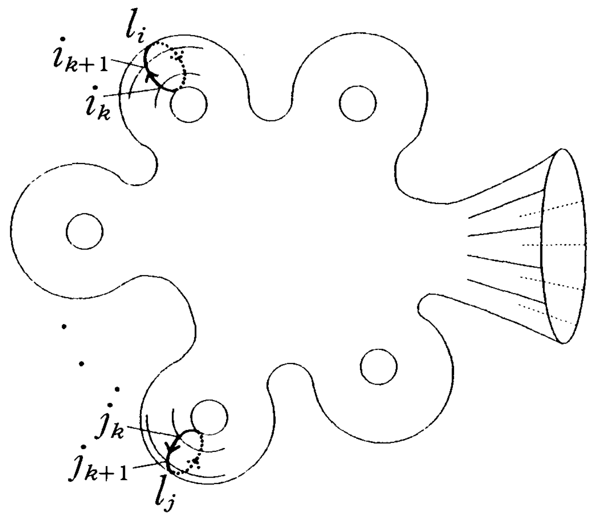


m_3
m_4
m_{g-2}
m_{g-1}
m_g
m_2
m_1
$\tilde{m}_1$
$\tilde{m}_2$
$\tilde{m}_3$
$\tilde{m}_{g-4}$
$\tilde{m}_{g-3}$
$\tilde{l}_1$
$\tilde{l}_2$
$\tilde{l}_3$
$\tilde{l}_{g-3}$
$\tilde{l}_{g-2}$
m_1
m_2
m_3
m_4
m_{g-2}
m_{g-1}
m_g

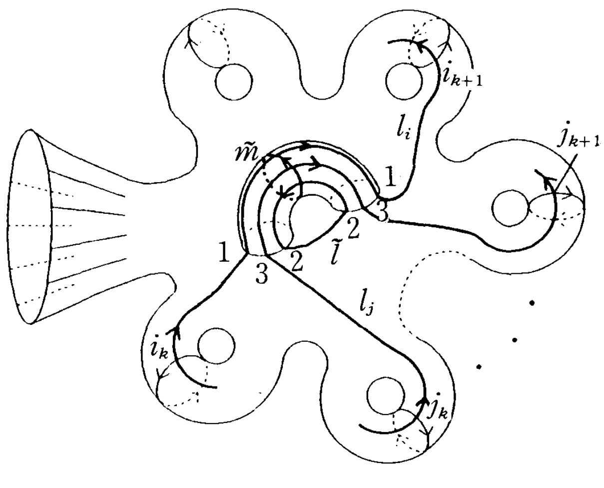
V6−B′



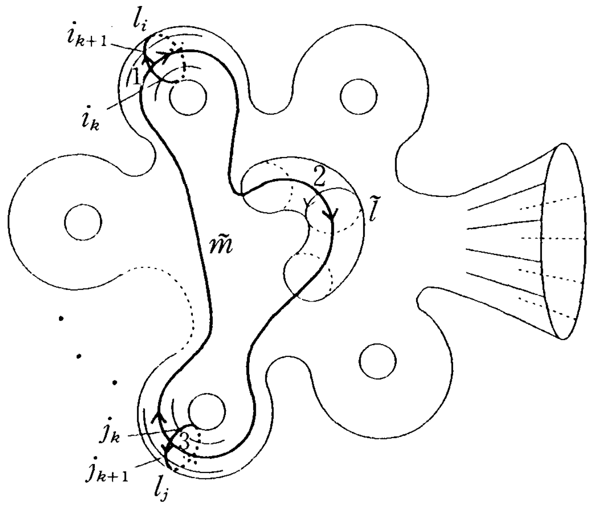
U7–A



V7–A′



U7–B



V7–B′

2. Connectedness of Heegaard diagrams and Heegaard cut diagrams

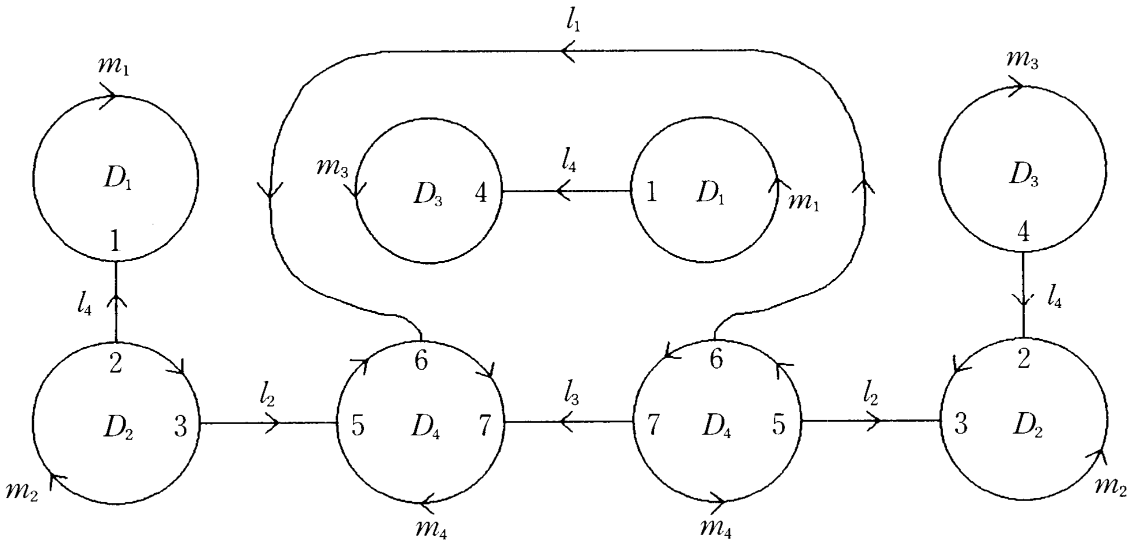
Let $(U\,;\,m,\,l) \cup (V\,;\,l,\,m)$ be genus n $(\geqq 2)$ H-diagrams of $(U,\,V,\,F)$ of M^3. $m = \{m_1,\,\cdots,\,m_n\}$ and $l = \{l_1,\,\cdots,\,l_n\}$. Let $G(m,\,l) \cup G(l,\,m)$ be the H-cut-diagrams of $(U\,;\,m,\,l) \cup (V\,;\,l,\,m)$, respectively. Let $G(m,\,l) \cup G(l,\,m)$ be the same presentation as in Def. 7 in [**10**, p. 45]. We choose $n-1$ edges from $\{j_k(l_j)j_{k+1}\}$ in $\{l_j\}(j = 1,\,\cdots,\,n)$ of $G(m,\,l)$ and let these edges be $\{L_1,\,\cdots,\,L_{n-1}\}$.

Definition 11. $(U\,;\, m,\, l)$ is called *connected* for the meridian system m if $\{L_1, \cdots, L_{n-1}\}$ can be chosen so that $(L_1 \cup \cdots \cup L_{n-1}) \cup (m_1 \cup \cdots \cup m_n)$ becomes a connected graph. If $(U\,;\, m,\, l)$ does not connect, then it is called *disconnected* for m. The closed connected orientable 3-manifolds, which have genus 1 H-diagram are 3-sphere S^3, lens spaces $L(p,\, q)$ and $S^2 \times S^1$. p and q of $L(p,\, q)$ are integers and relatively prime. We define that each genus 1 H-diagram of S^3 and $L(p,\, q)$ is connected and that of $S^2 \times S^1$ is disconnected for $m = \{m_1\}$.

Definition 12. If the closures of connected components of $S_l^2 - |G(m,\, l)|$ consist of 2-disks, then $G(m,\, l)$ is called *connected for* m *and* l. If $G(m,\, l)$ is not connected, then it is called *disconnected for* m *and* l.

Fig. 1 shows a disconnected H-cut-diagram $G(m,\, l)$ of $(U\,;\, m,\, l)$ of $S^2 \times S^1$ but here the connection of H-diagram $(U\,;\, m,\, l)$ is made.

Therefore, we see the following obvious proposition.



$G(m,\, l)$

Fig. 1

Proposition 2. *The connectedness of H-cut-diagram* $G(m,\, l)$ *is equivalent to that of* $G(l,\, m)$. *If* $G(m,\, l)$ *of H-diagram* $(U\,;\, m,\, l)$ *is connected, then* $(U\,;\, m,\, l)$ *also becomes connected but the reverse of this does not hold generally. If H-genus* $= 1$, *then the connectedness of* $(U\,;\, m,\, l)$ *is equivalent to that of* $G(m,\, l)$ *where* $m = \{m_1\}$ *and* $l = \{l_1\}$.

3. Transformations of Heegaard cut diagrams

From now on, we give transformations of H-cut-diagrams besides those given in [**10**].

Definition 13.

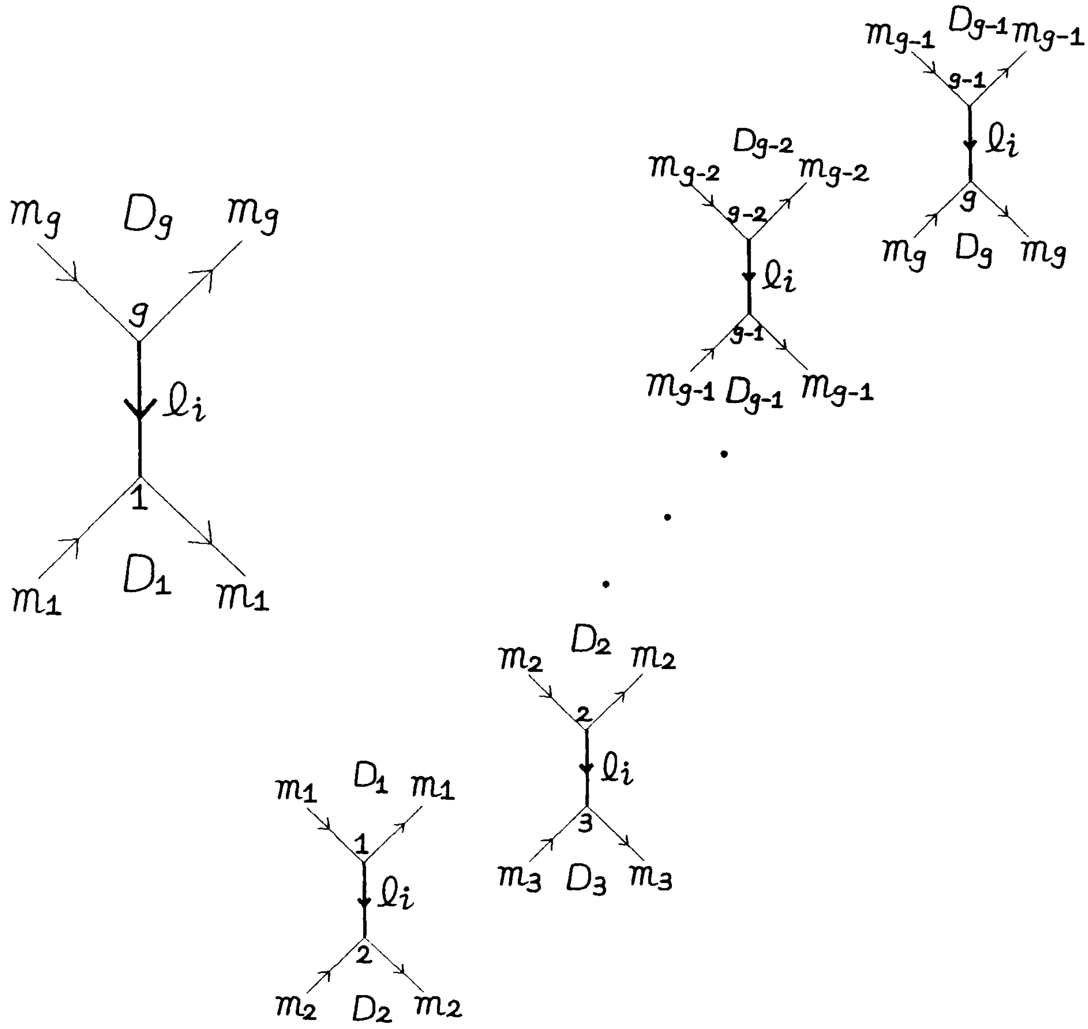
Table 6[4]. H_{bs}, H_e, H_d, H_t-*transformation*

6–A ⇒ 6–B	$H_{bs}(g+1^+, (g-2)3^+)$	$g \geqq 4$
6–B ⇒ 6–A	$H_e((g-3)2^-)$	
6–A′ ⇒ 6–B′	$H_d(3^-, (g-2)3^+)$	
6–B′ ⇒ 6–A′	$H_t((g-3)2^-)$	

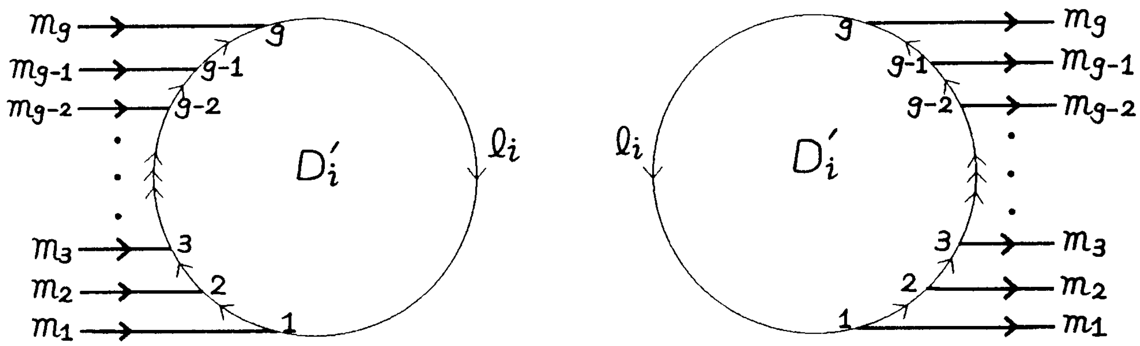
6 – A, 6 – A′, 6 – B, 6 – B′ denotes the following figure, i.e., the part of a H-cut-diagram, without dotted lines, respectively. Later dotted lines are made use of transformations of H-cut-diagrams. 6–A′, 6–B′ is the dual part to 6–A, 6–B, respectively.

Transformation from 6–A (6–A′ resp.) to 6–B (6–B′ resp.) increases the H-genus as many as $g-3$ ($g \geqq 4$), the cross point number as many as $2g-6$.

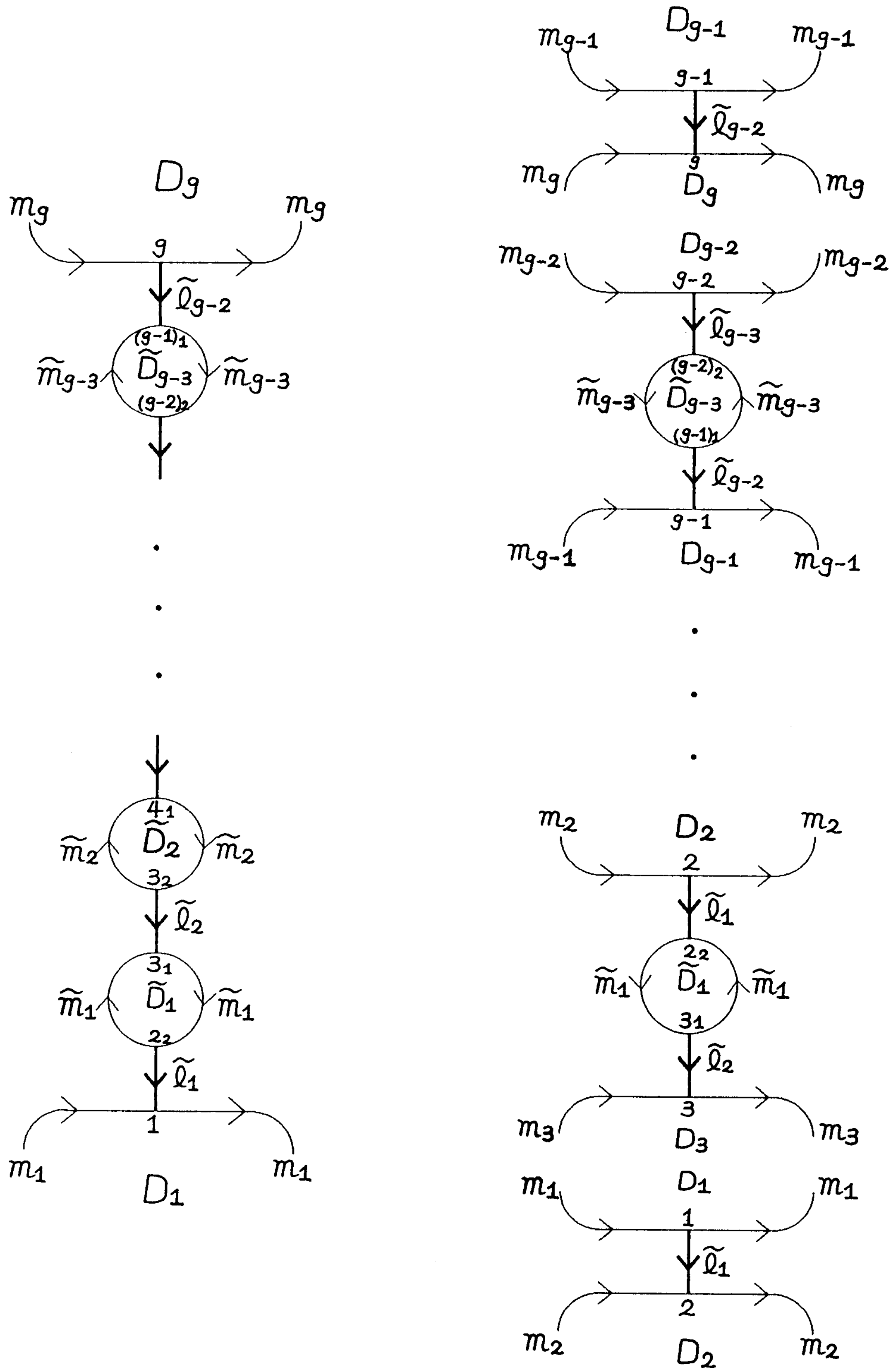
[4] Table 6 is a serial notation from Table 5 in [10].


m_g D_g m_g
g
ℓ_i
1
m_1 D_1 m_1
m_{g-1} D_{g-1} m_{g-1}
$g-1$
m_{g-2} D_{g-2} m_{g-2}
$g-2$
$g-1$
m_2 D_2 m_2
2
3
m_3 D_3 m_3

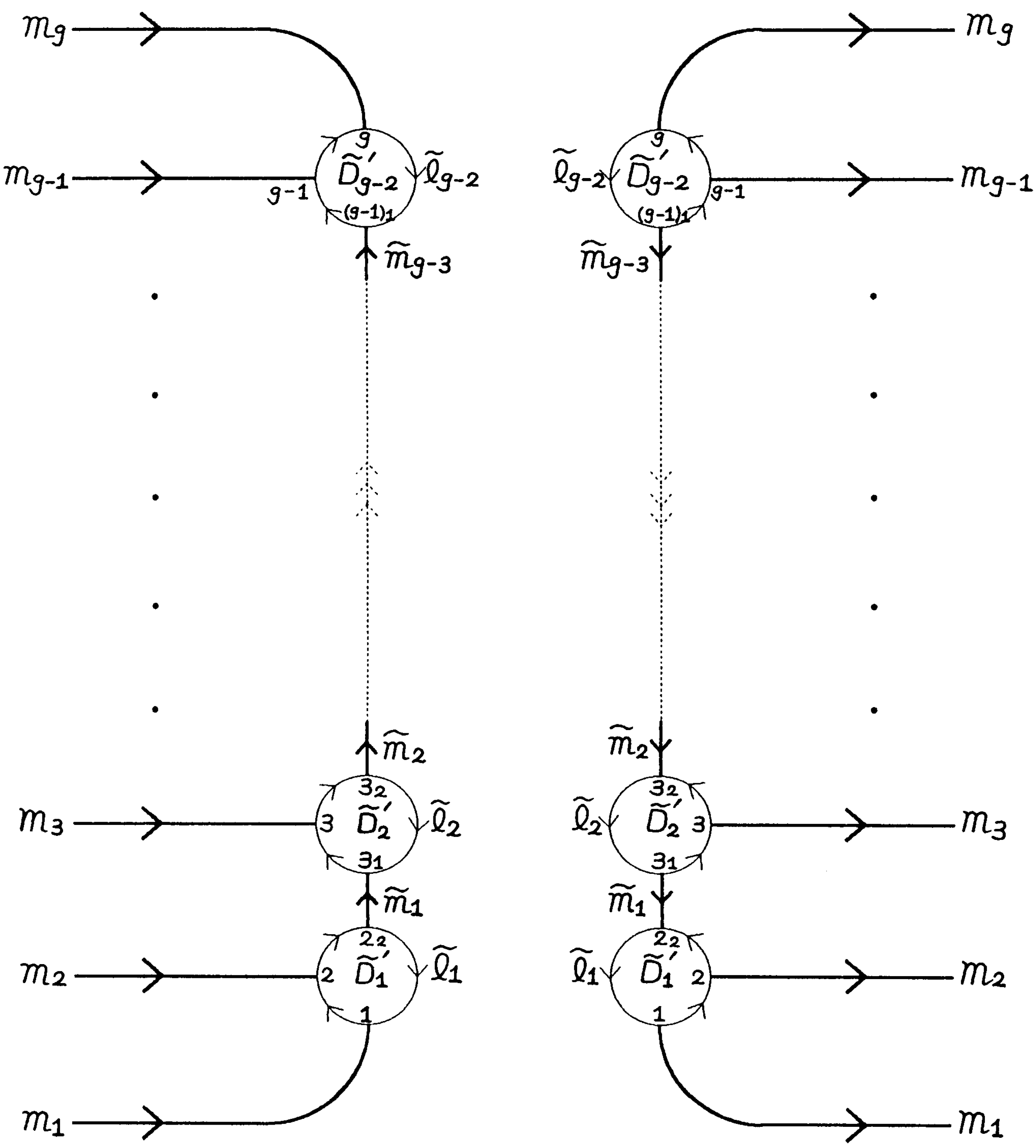
6—A


m_g
m_{g-1}
m_{g-2}
m_3
m_2
m_1
g
$g-1$
$g-2$
3
2
1
D_i'
ℓ_i

6—A′



6—B



6—B′

Let $(K\ ;\ m,\ l)$, $(L\ ;\ l,\ m)$ be the canonical genus g ($\geqq$ 1) H-diagram of $(K,\ L,\ T_g)$ of S^3 and $G_K(m,\ l)$, $G_L(l,\ m)$ the H-cut-diagram, respectively. Let $(U\ ;\ m,\ l)$, $(V\ ;\ l,\ m)$ be a genus n ($\geqq$ 1) H-diagram of $(U,\ V,\ F)$ of M^3 and $G(m,\ l)$, $G(l,\ m)$, the H-cut-diagram, respectively. If we construct the connected sum $M^3 \# S^3$ [5], then a corresponding H-splitting is naturally obtained. Let the notation be $(K,\ L,\ T_g)\#(U,\ V,\ F)$. Let H-diagrams of $(K,\ L,\ T_g)\#(U,\ V,\ F)$ be $(K\ ;\ m,\ l)\#(U\ ;\ m,\ l)$ and $(L\ ;\ l,\ m)\#(V\ ;\ l,\ m)$, respectively. Let the H-cut-diagrams be $G_K(m,\ l)\#G(m,\ l)$, $G_L(l,\ m)\#G(l,\ m)$, respectively. Then $G_K(m,\ l)\#G(m,\ l)$ ($G_L(l,\ m)\#G(l,\ m)$ resp.) becomes genus $n+g$ disconnected H-cut-diagram of $(K,\ L,\ T_g)\#(U,\ V,\ F)$ of M^3. Then we have ;

[5] $M^3 \# S^3$ is homeomorphic to M^3.

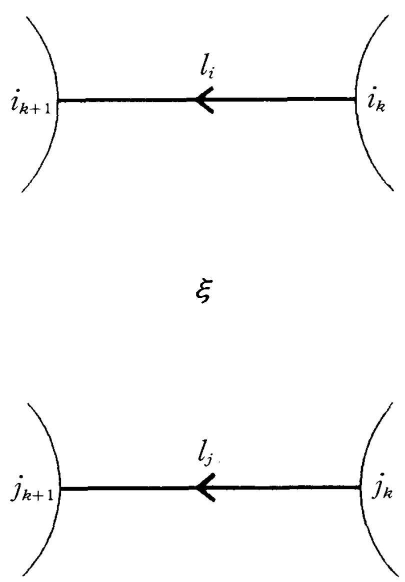
Table 7. *$H^{\#}$-transformation*

$G(m, l) \Rightarrow G_K(m, l) \# G(m, l)$	$H^{\#}(g^{+}(S^3, T_g))$
$G(l, m) \Rightarrow G_L(l, m) \# G(l, m)$	
$G_K(m, l) \# G(m, l) \Rightarrow G(m, l)$	$H^{\#}(g^{-}(S^3, T_g))$
$G_L(l, m) \# G(l, m) \Rightarrow G(l, m)$	

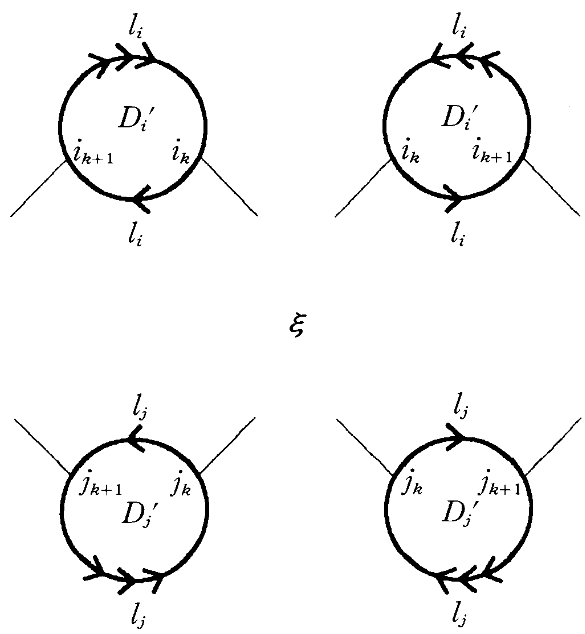
The transformations of $H^{\#}(g^{+}(S^3, T_g))$ increase the H-genus and the cross point number as many as g.

Table 8. *H_t, H_b-transformation*

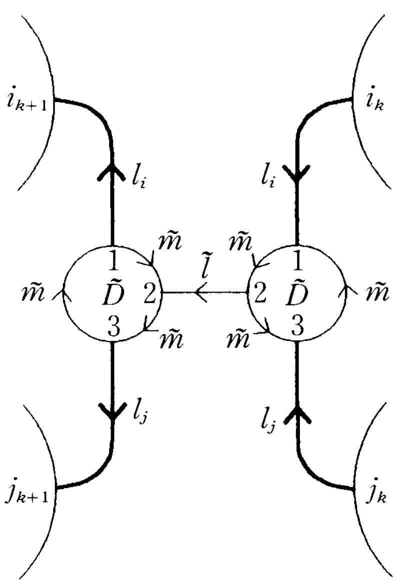
7–A $\Rightarrow$ 7–B	$H_t(2^+, 3^+)$
7–B $\Rightarrow$ 7–A	$H_t(3^-, 2^-)$
7–A′ $\Rightarrow$ 7–B′	$H_b(2^+, 3^+)$
7–B′ $\Rightarrow$ 7–A′	$H_b(3^-, 2^-)$



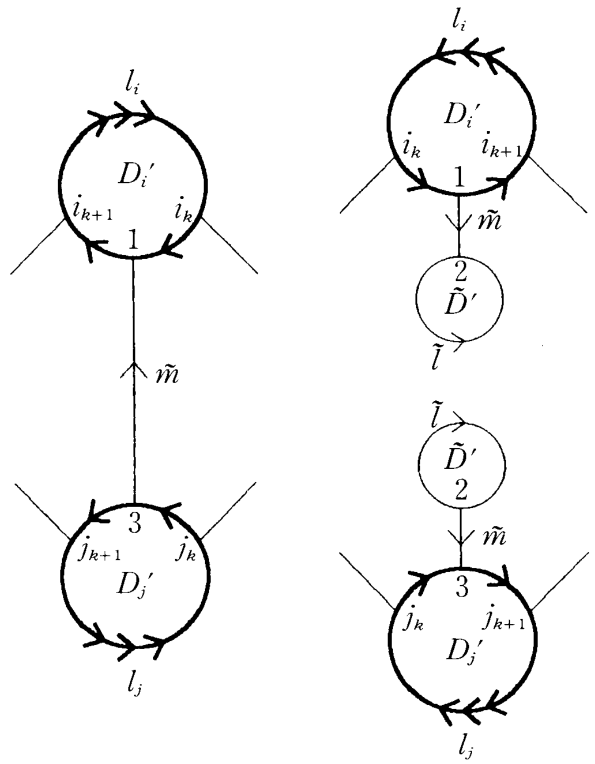
7—A



7—A′



7—B

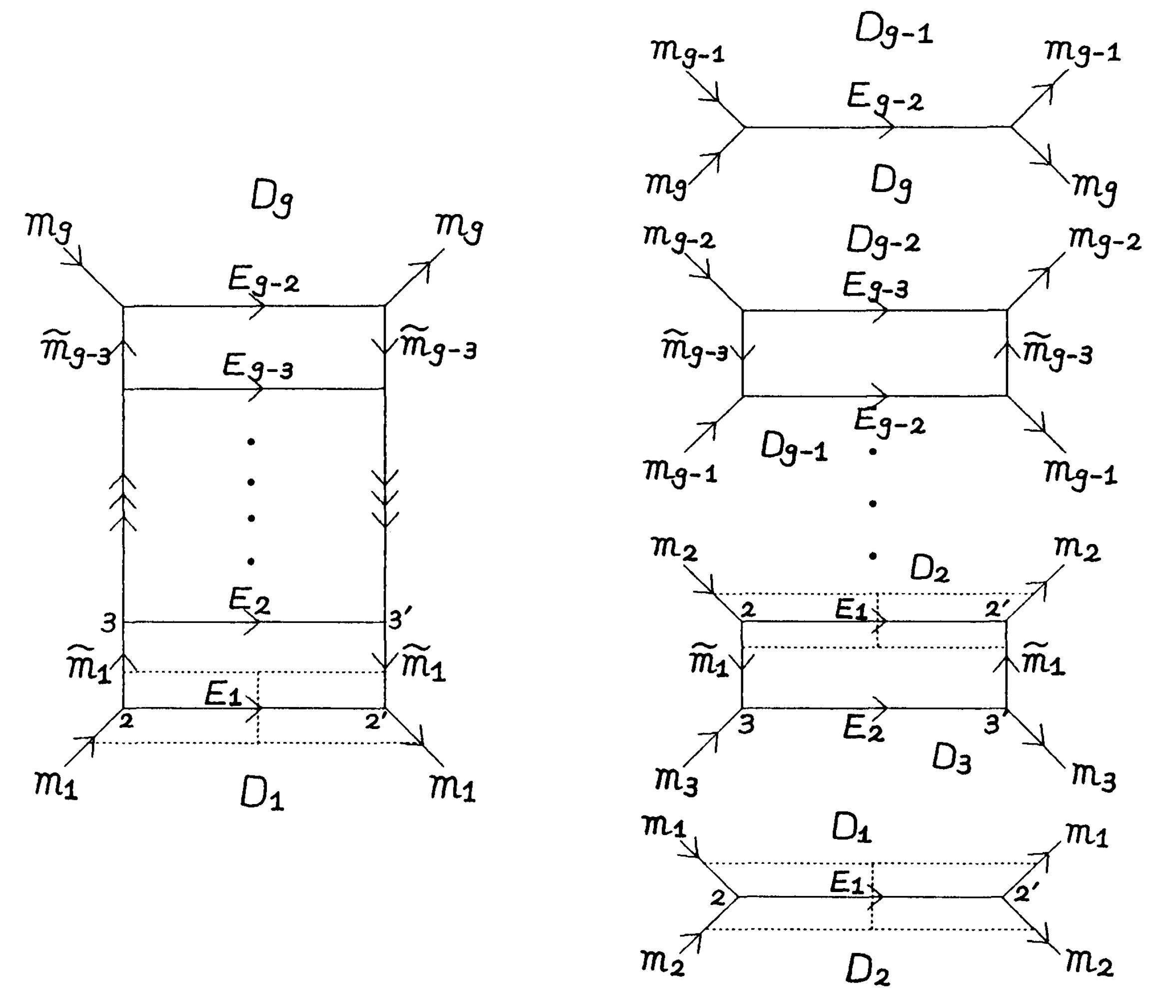


7—B′

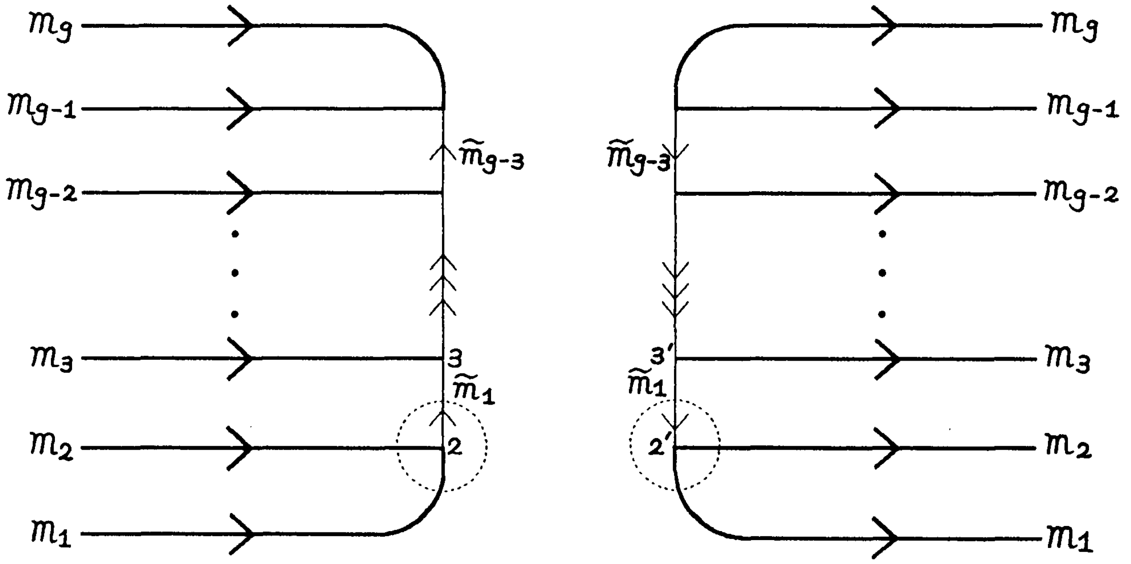
Transformation from 7－A (7－A′ resp.) into 7－B (7－B′ resp.) increases the H-genus as many as 1, the cross point number as many as 3.

Next we illustrate the methods of the above H-cut-transformations using the DS-deformations [**4**].

$H_{bs}(g+1^{+}, (g-2)3^{+})$, $H_d(3^{-}, (g-2)3^{+})$-transformation : 6－A $\Rightarrow$ 6－B and 6－A′ $\Rightarrow$ 6－B′
Step 1. Applying $D_{g+1}{}^{+}$-deformation to (6－A) $\cup$ (6－A′), the following figures (6－A1) $\cup$ (6－A1′) are obtained.

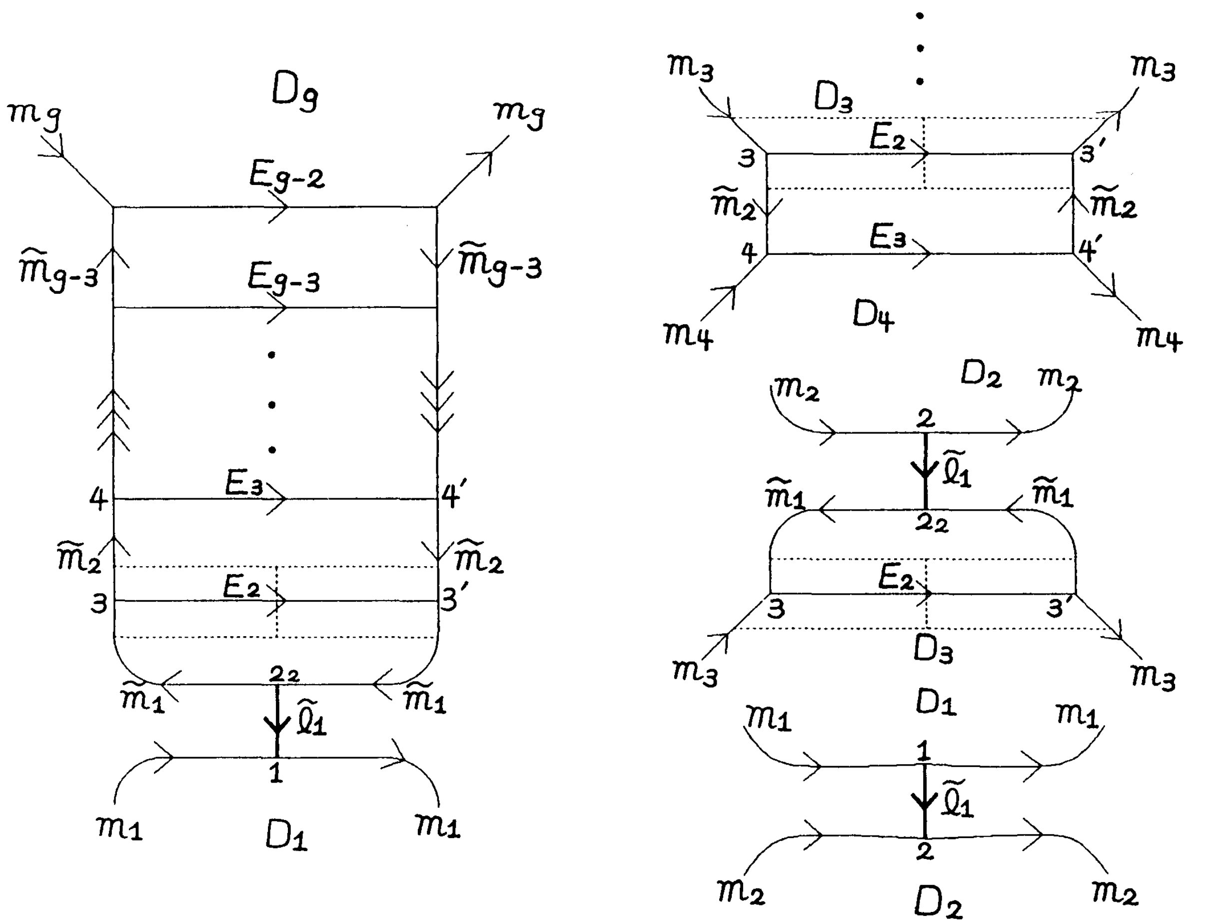

D_g
m_g
E_{g-2}
$\widetilde{m}_{g-3}$
E_{g-3}
E_2
$\widetilde{m}_1$
E_1
m_1
D_1
D_{g-1}
m_{g-1}
m_{g-2}
D_{g-2}
m_2
D_2
m_3
D_3
E_2

6—A1

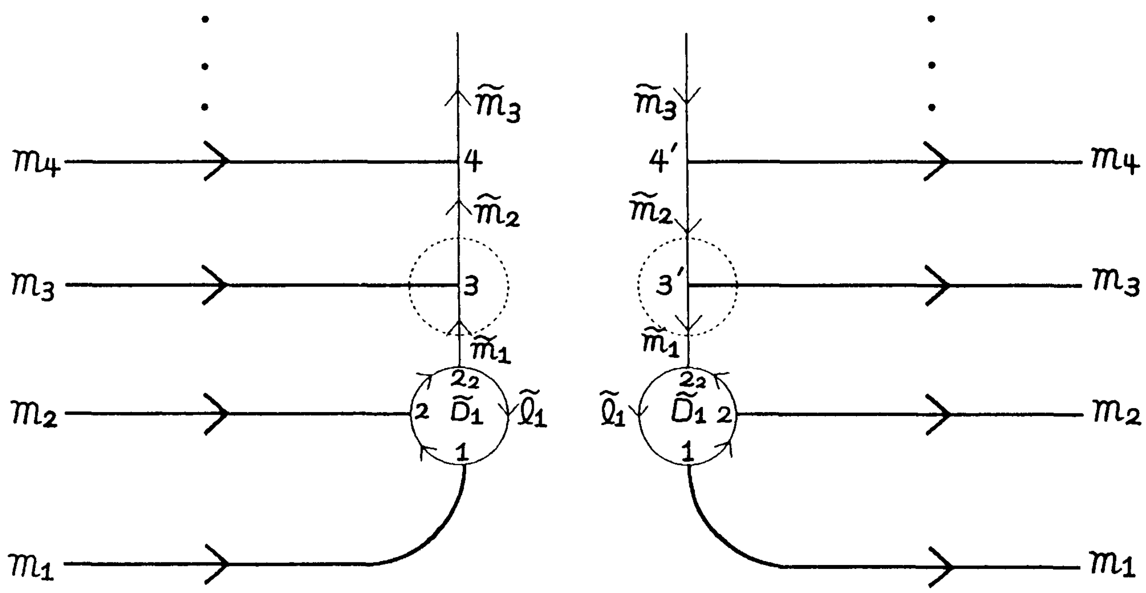

m_g
m_{g-1}
$\widetilde{m}_{g-3}$
m_{g-2}
m_3
$\widetilde{m}_1$
m_2
m_1

6—A1′

Step 2.　Applying $D_3{}^+$-deformation to (6–A1) ∪ (6–A1′) shown as the six dotted lines and two circles, (6–A2) ∪ (6–A2′) are obtained.

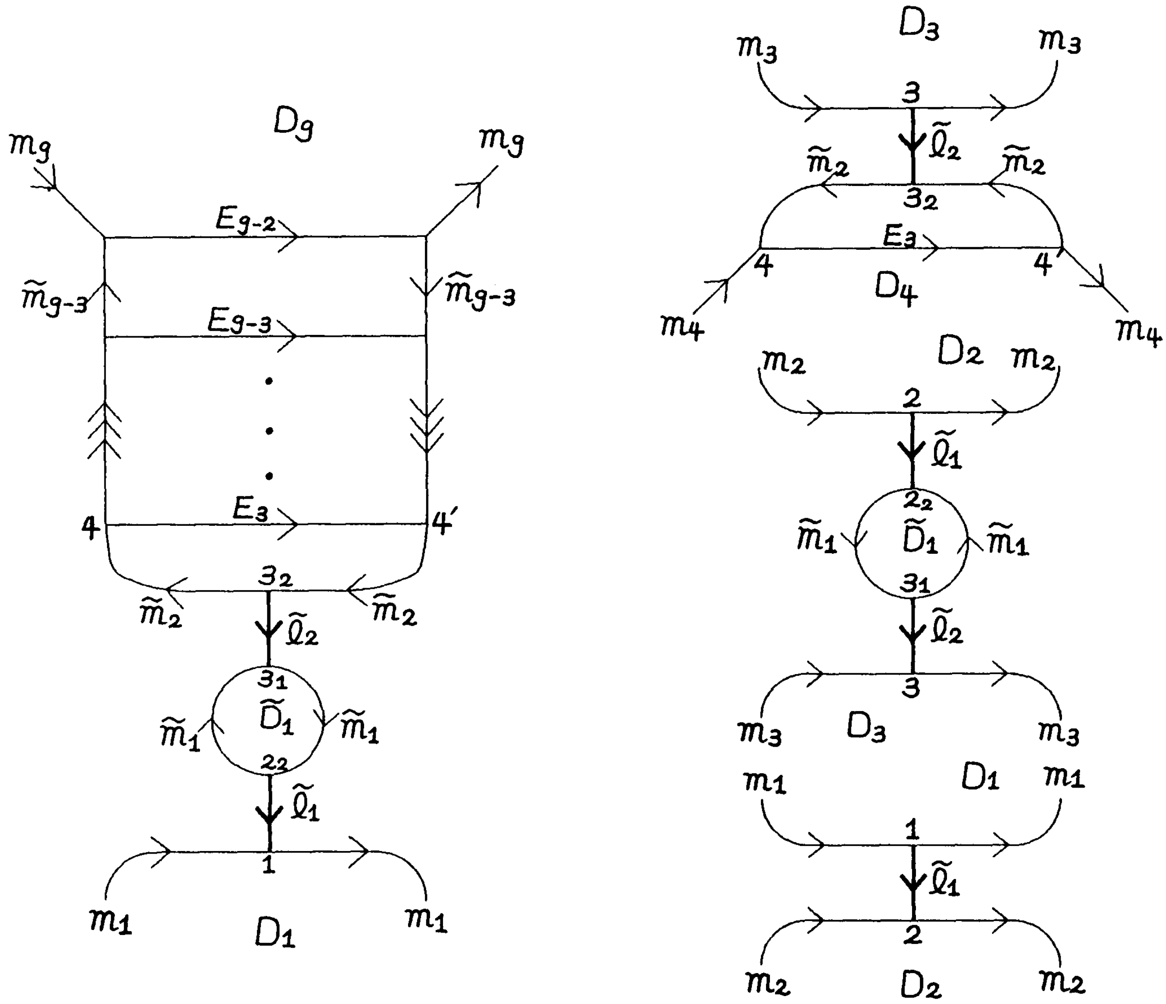


6–A2

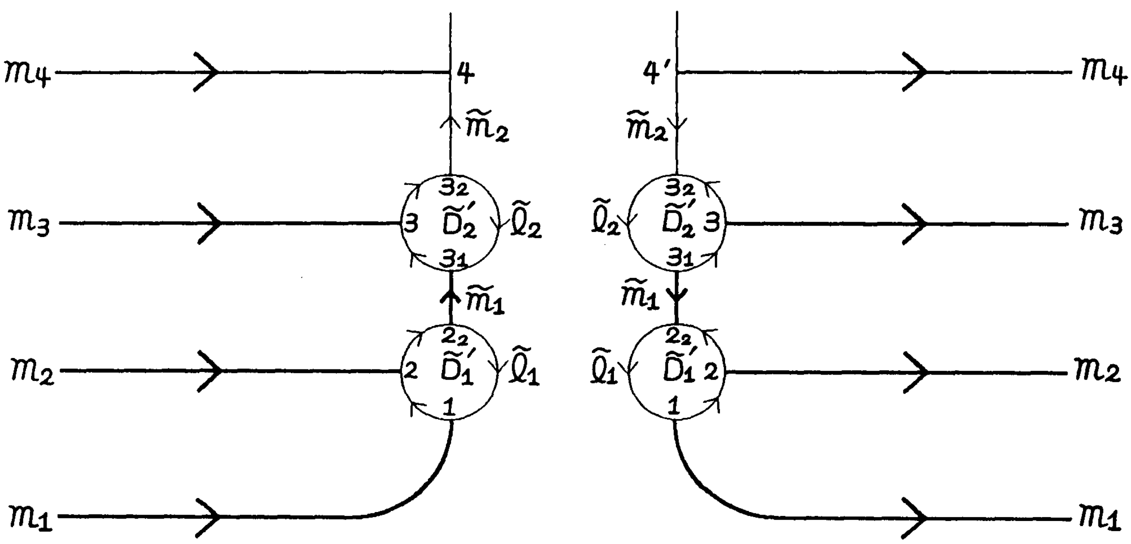


6–A2′

Step 3. Again applying $D_3{}^+$-deformation to (6–A2) ∪ (6–A2′) shown as the dotted lines and circles, (6–A3) ∪ (6–A3′) are obtained.



6–A3



6–A3′

Step 4. Similarly we continue to keep $D_3{}^+$-deformation from (6−A3) ∪ (6−A3′) so that edge E_i ($i = 3, \cdots, g-2$) is erased, then (6−B) ∪ (6−B′) are obtained.

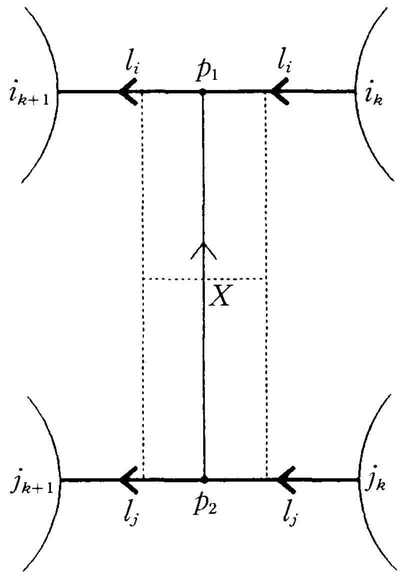
To make a transformation from 6−B (6−B′ resp.) into 6−A (6−A′ resp.), we may carry out the finite sequence of $D_2{}^-$-deformations so that those erase the disks $\{\tilde{D}_i\}$ in 6−B.

H_t, H_b-transformation : 7−A ⇒ 7−B and 7−A′ ⇒ 7−B′

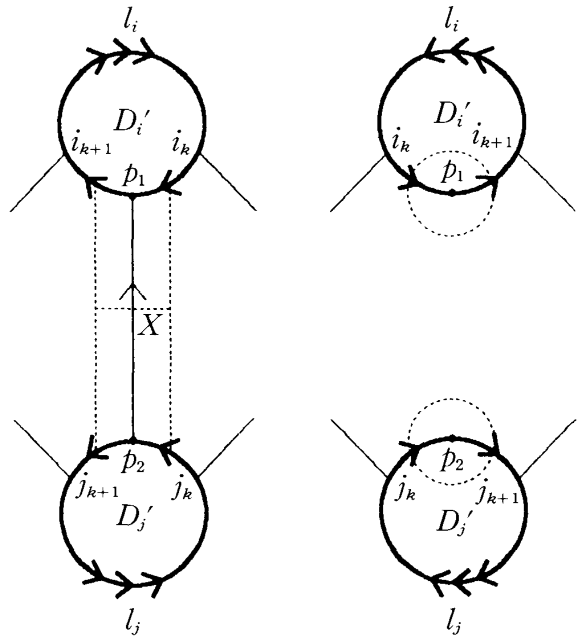
Step 1. Let ξ be a face (disk or punctured 2-disk[6]) in the closures of connected components of $S_{l'}{}^2 - |G(m, l)| - \cup_{j=1}^{2} (D_j \cup D_j)$. ξ satisfies the following condition.

(C) $i_k(l_i)i_{k+1} \subset \partial\xi$ and $j_k(l_j)j_{k+1} \subset \partial\xi$ $(i \neq j)$

Step 2. There exists the same labeled face ξ in $G(l, m)$ that is inversely oriented to that in $G(m, l)$. Draw the same oriented labeled edges X in both ξ so that $p_1 \subset Int(i_k(l_i)i_{k+1}) \subset \partial\xi$ and $p_2 \subset Int(j_k(l_j)j_{k+1}) \subset \partial\xi$ where $\{p_1, p_2\}$ are two points of ∂X. 7−A1 (7−A1′ resp.) denotes $\{G(m, l) \cup X\}(\{G(l, m) \cup X\}$ resp.).



7—A1

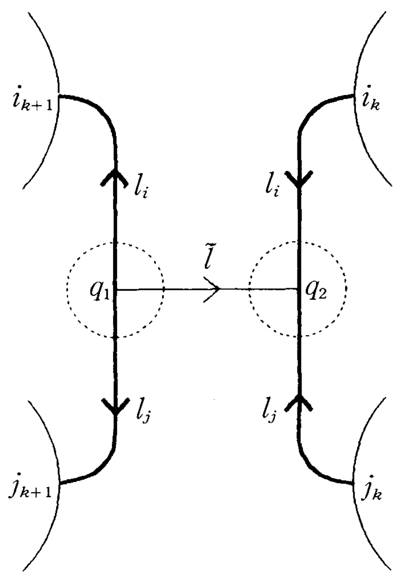


7—A1′

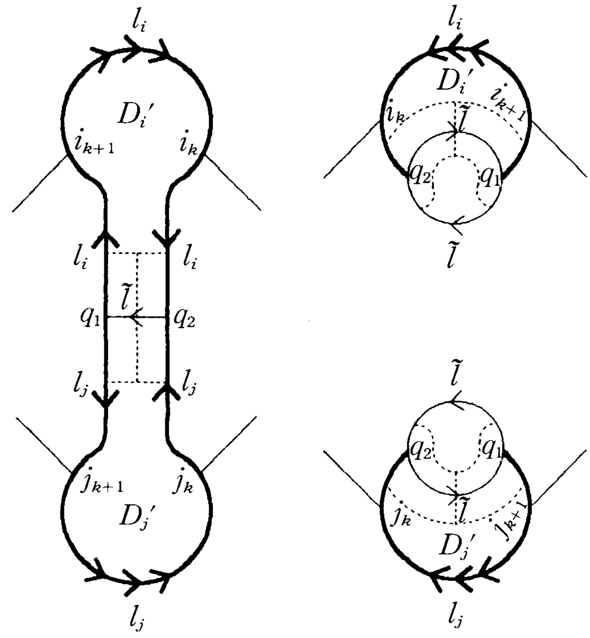
---

[6] disk with n (≧ 1) holes.

Step 3. Carry out D_2^+-deformation shown as the four dotted lines and two circles in $\{G(m, l) \cup X\} \cup \{G(l, m) \cup X\}$, then deformed diagrams 7–A2 and 7–A2′ are obtained.



7—A2



7—A2′

Step 4. Carry out D_3^+-deformation to $q_1(\tilde{l})q_2$ shown as the dotted lines and circles in (7–A2) $\cup$ (7–A2′), then (7–B) $\cup$ (7–B′) are obtained.

Remark 1. In (C), we may change $\{i_k(m_i)i_{k+1},\ j_k(m_j)j_{k+1}\}(i \neq j)$ instead of $\{i_k(l_i)i_{k+1},\ j_k(l_j)j_{k+1}\}$ but if we take a different pair $\{i_k(l_i)i_{k+1},\ i_k(m_i)i_{k+1}\}$ of a longitude and meridian instead of $\{i_k(l_i)i_{k+1},\ j_k(l_j)j_{k+1}\}$, then H-cut-diagrams can not be constructed from $\{G(m, l) \cup X\} \cup \{G(l, m) \cup X\}$.

Remark 2. The diagrams shown in (7 – A1) $\cup$ (7 – A1′)((7 – A2) $\cup$ (7 – A2′) resp.) are not H-cut-diagrams but they represent M^3.

We consider in the case that ξ does not satisfy the condition (C). Let such ξ be η. Let $(K\,;\, m, l)$, $(L\,;\, l, m)$ be the canonical genus 1 H-diagram of (K, L, T_1) of S^3 and $G_K(m, l)$, $G_L(l, m)$ the H-cut-diagram, respectively. We draw $G_K(m, l)$ ($G_L(l, m)$ resp.) in $Int(\eta)$ ($\subset |G(m, l)|$) ($Int(\eta)$ ($\subset |G(l, m)|$) resp.). Then $G_K(m, l) \cup G(m, l)$ ($G_L(l, m) \cup G(l, m)$ resp.) becomes a H-cut-diagram $G_K(m, l) \# G(m, l)$ ($G_L(l, m) \# G(l, m)$ resp.) of $(K\,;\, m, l) \# (U\,;\, m, l)$ ($(L\,;\, l, m) \# (V\,;\, l, m)$ resp.) of connected sum $M^3 \# S^3$. From $G_K(m, l) \cup \eta$ and $G_L(l, m) \cup \eta$ we may find the faces such as ξ in (7 – A) $\cup$ (7 – A′). Therefore carrying out the same transformations as in the above (step 1∼4), the parts of (7 – B) $\cup$ (7 – B′) that increase the H-genus as many as 2 are obtained from $(G_K(m, l) \# G(m, l)) \cup (G_L(l, m) \# G(l, m))$.

Using the above transformations of (7–A) $\cup$ (7–A′) $\Rightarrow$ (7–B) $\cup$ (7–B′) repeatedly, we have ;

Proposition 3. *Let* $G(m,\ l)$ ($G(l,\ m)$ *resp.*) *be a disconnected genus* n ($\geqq 1$) *H-cut-diagram of* $(U,\ V,\ F)$ *of* M^3. *Then there exists a transformation from the disconnected H-cut-diagram into connected one.*

Proof. We give an algorithm of construction as follows.

Step 1. Since $G(m,\ l)$ is disconnected, there exists a punctured 2-disk in $G(m,\ l)$. Let ξ be punctured 2-disks in both $G(m,\ l)$ and $G(l,\ m)$. If ξ does not satisfy the condition (C), then we construct the connected sum $M^3 \# S^3$ such as given in front of this Prop.

Step 2. We draw the same oriented labeled edges $X_1,\ \cdots,\ X_t$ in both ξ that satisfy the following conditions:

(1) Let $\{p_{i_1},\ p_{i_2}\}$ be two points of ∂X_i and $\{p_{i_1},\ p_{i_2}\}$ ($i = 1,\ \cdots,\ t$) differently.

(2) $p_{i_1} \subset Int(j_{i_1}(l_{j_{i_1}})j_{i_1+1}) \subset \partial\xi$ and $p_{i_2} \subset Int(j_{i_2}(l_{i_2})j_{i_2+1}) \subset \partial\xi$.

(3) The closure of the connected component of $\xi-(X_1 \cup \cdots \cup X_t)$ becomes 2-disk.

Step 3. Carry out the transformations from (7–A) ∪ (7–B) into (7–A′) ∪ (7–B′) to both parts of the X_1 in $\{G(m,\ l) \cup X_1\} \cup \{G(l,\ m) \cup X_1\}$. Then we get genus $n+1$ or $n+2$ H-cut-diagrams.

Step 4. Carry out the same operations as step 3 to other X_i ($i = 2,\ \cdots,\ t$), then we obtain H-cut-diagrams in which genuses are more than $n+t-1$. Let the H-cut-diagrams be $G'(m,\ l) \cup G'(l,\ m)$.

Step 5. If $G'(m,\ l)$ is not connected, carry out the same operations from step 1 to 4 to $G'(m,\ l) \cup G'(l,\ m)$, repeatedly, then connected H-cut-diagrams are obtained. □

Let $(U_i\,;\ m,\ l)$, $(V_i\,;\ l,\ m)$ ($i = 1, 2$) be genus n_i ($\geqq 1$) H-diagram of $(U_i,\ V_i,\ F_i)$ of M_i^3 and $G_{U_i}(m,\ l)$, $G_{V_i}(l,\ m)$ the H-cut-diagram, respectively. If we construct the connected sum $M_1{}^3 \# M_2{}^3$, then we get disconnected genus n_1+n_2 H-diagrams $(U_1\,;\ m,\ l) \# (U_2\,;\ m,\ l)$, $(V_1\,;\ l,\ m) \# (V_2\,;\ l,\ m)$ of $(U_1,\ V_1,\ F_1) \# (U_2,\ V_2,\ F_2)$ and their disconnected H-cut-diagrams $G_{U_1}(m,\ l) \# G_{U_2}(m,\ l)$, $G_{V_1}(l,\ m) \# G_{V_2}(l,\ m)$. Then we have;

Proposition 4. *There exists a transformation from the disconnected H-cut-diagram* $G_{U_1}(m,\ l) \# G_{U_2}(m,\ l)$ ($G_{V_1}(l,\ m) \# G_{V_2}(l,\ m)$ *resp.*) *of* $(U_1,\ V_1,\ F_1) \# (U_2,\ V_2,\ F_2)$ *of* $M_1{}^3 \# M_2{}^3$ *into connected one.*

4. Proofs of Theorem 3

Proofs of (7) *and* (7′). The H-cut-diagrams derived from U6–A (V6–A′ resp.) and U6–B (V6–B′ resp.) are 6–A (6–A′ resp.) and 6–B (6–B′ resp.), respectively. Therefore the transformation from 6–A (6–A′ resp.) into 6–B (6–B′ resp.) gives a transformation from U6–A (U6–A′ resp.) into U6–B (U6–B′). □

Proofs of (8) *and* (8′). The transformation from 7–A (7–A′ resp.) into 7–B (7–B′ resp.) gives a transformation from U7–A (U7–A′ resp.) into U6–B (U6–B′). □

Definition 14. The transformations of theorem 2 ([**10**]) and 3 will be generally called the *H-transformations for Heegaard diagrams*.

Relations between the operations used for the transformations of H-diagrams and DS-deformations used for those of H-cut-diagrams are a theorem stated as the following table.

Theorem 4.

<table>
<tr><th>Transformation of H-diagram</th><th>Operation for H-diagram</th><th>Transformation of H-cut-diagram</th><th>DS-deformation</th></tr>
<tr><td>U6–A $\Rightarrow$ U6–B</td><td>or $g-3$ handles adding handlebody cutting by $g-3$ annuluses</td><td>6–A $\Rightarrow$ 6–B</td><td rowspan="2">${D_{g+1}}^{+} \to$ $g-2$ times of ${D_3}^{+}$</td></tr>
<tr><td>V6–A′ $\Rightarrow$ V6–B′</td><td>handle cutting and $g-2$ handles adding</td><td>6–A′ $\Rightarrow$ 6–B′</td></tr>
<tr><td>U7–A $\Rightarrow$ U7–B</td><td>or handle adding handlebody cutting by an annulus</td><td>7–A $\Rightarrow$ 7–B</td><td rowspan="2">${D_3}^{+} \to {D_3}^{+}$</td></tr>
<tr><td>V7–A′ $\Rightarrow$ V7–B′</td><td>or handle adding handlebody cutting by an annulus</td><td>7–A′ $\Rightarrow$ 7–B′</td></tr>
</table>

Further we derive the following H-transformations from Prop. 3, 4, respectively.

Proposition 5. *Let $(U\,;\,m,\,l)$ $((V\,;\,l,\,m)$ resp.) be a disconnected genus n $(\geqq 1)$ H-diagram of $(U,\,V,\,F)$ of M^3. Then there exists a transformation from the disconnected H-diagram into connected one.*

Proposition 6. *There exists a transformation from a disconnected H-diagram $(U_1\,;\,m,\,l)\,\#\,(U_2\,;\,m,\,l)$ $((V_1\,;\,l,\,m)\,\#\,(V_2\,;\,l,\,m)$ resp.) of $(U_1,\,V_1,\,F_1)\,\#\,(U_2,\,V_2,\,F_2)$ of $M_1{}^3\,\#\,M_2{}^3$ into connected one.*

5. Some examples

Example 4. We begin with the definition of a wave.

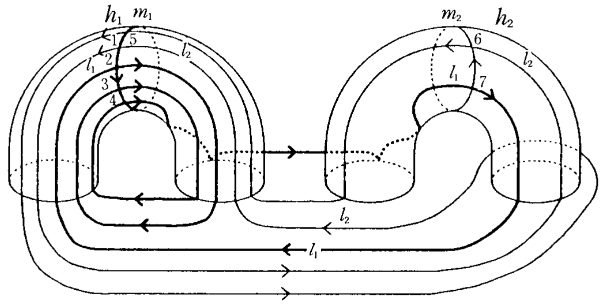
Definition 15. Let σ_i be a disk in the closures of connected components of $S_U{}^2 - |G(m, l)| - \cup_{j=1}^{n}(D_j \cup D_j)$. $\partial\sigma_i$ consists of a part of edges in $G(m, l)$. Suppose that there exist oriented two edges $j_\alpha(m_j)j_{\alpha+1}$, $j_\beta(m_j)j_{\beta+1}(j_\alpha(l_j)j_{\alpha+1}$, $j_\beta(l_j)j_{\beta+1}$ resp.) $(\alpha \neq \beta)$ of $m_j(l_j$ resp.) in $\partial\sigma_i$. When we go round the circle $\partial\sigma_i$ clockwise or counterclockwise, suppose that the orientations of both $j_\alpha(m_j)j_{\alpha+1}$ and $j_\beta(m_j)j_{\beta+1}$ ($j_\alpha(l_j)j_{\alpha+1}$ and $j_\beta(l_j)j_{\beta+1}$ resp.) become the same as that. Let w be an arc in σ_i and ∂w points p_1, p_2 such as $p_1 \subset Int(j_\alpha(m_j)j_{\alpha+1})$ $(Int(j_\alpha(l_j)j_{\alpha+1})$ resp.) and $p_2 \subset Int(j_\beta(m_j)j_{\beta+1})$ $(Int(j_\beta(l_j)j_{\beta+1})$ *resp.). Then* w is called *a wave of* $G(m, l)$. Similarly a wave of $G(l, m)$ is defined.

By the definition of a wave, if $G(m, l)$ has waves $\{w_1, w_2, \cdots, w_\alpha\}$, then $G(l, m)$ has the same waves as $G(m, l)$.

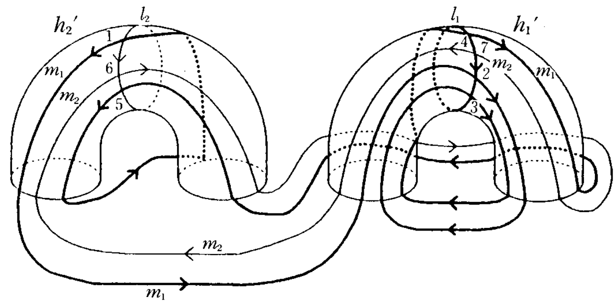
The Whitehead [**11**]-Volodin-Kuznetsov-Fomenko [**12**] conjecture shows that "all H-diagrams of the 3-sphere S^3 other than the canonical one have waves without fail." This is an algorithm for recognizing S^3 in 3-manifold. In [**14**], Birman says that "nobody has succeeded in verifying such an assertion between 1935 and 1977, or producing a counter example." In 1980, Homma-Ochiai-Takahashi [**17**] success in the above conjecture if H-genus $= 2$ but Viro [**13**], Morikawa [**18**] and Ochiai [**19**] construct counter examples if H-genus $\geqq 3$.

A counter example by a transformation from waves H-cut-diagram into no-waves one is obtained in [**9**, p. 76]. Fig. 2 gives genus 2 H-diagram $(U\,;\, m, l, 7)$, $(V\,;\, l, m, 7)$ of S^3 and those H-cut-diagrams $G_1(m, l, 7)$, $G_1(l, m, 7)$, respectively. Here $U = B_U{}^3 + (h_1 \cup h_2)$ and $V = B_V{}^3 + (h_1' \cup h_2')$. The number 7 of $(U\,;\, m, l, 7)$ ($G_1(m, l, 7)$ resp.) indicates the cross point number of $m \cap l$. $G_1(m, l, 7)$, $G_1(l, m, 7)$ has two waves drawn by wavy lines, respectively. Here $Cl(\partial B_U{}^3 - \cup_{i=1}^{2}\{D_i \times 0,\ D_i \times 1\})$ $(Cl(\partial B_V{}^3 - \cup_{i=1}^{2}\{D_i' \times 0,\ D_i' \times 1\})$ resp.) is the H-cut-diagram $G_1(m, l, 7)$ ($G_1(l, m, 7)$ resp.).

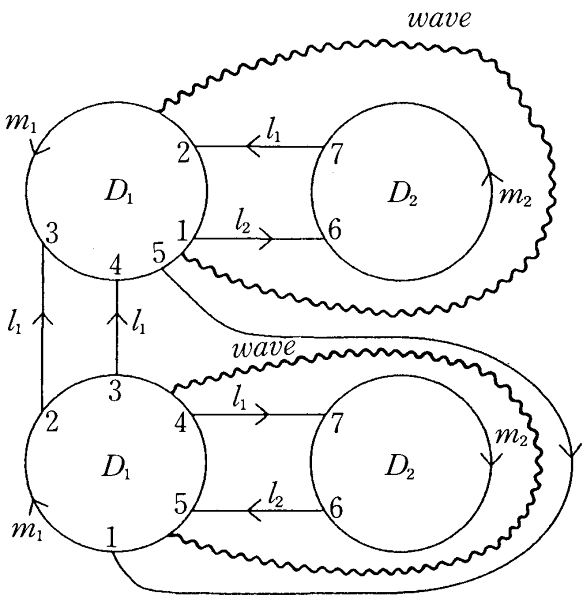
From now on we carry out transformations of the above H-diagrams $(U\,;\, m, l, 7) \cup (V\,;\, l, m, 7)$ corresponding to those from waves H-cut-diagrams $G_1(m, l, 7) \cup G_1(l, m, 7)$ into no-waves ones given in [**9**].



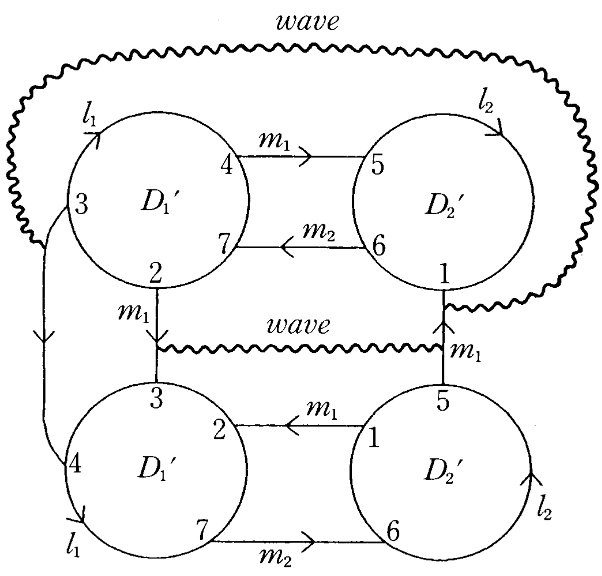
(U ; m, l, 7)



(V ; l, m, 7)



$G_1(m, l, 7)$

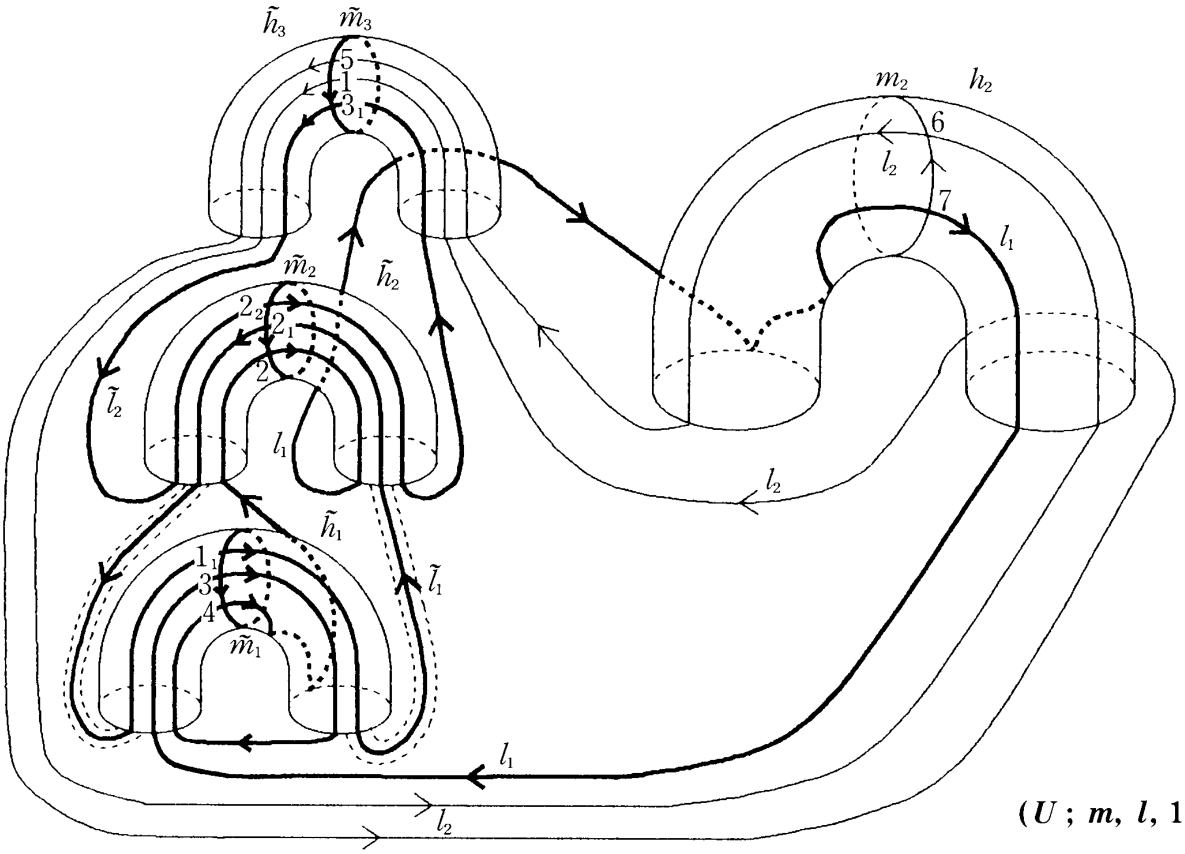


$G_1(l, m, 7)$

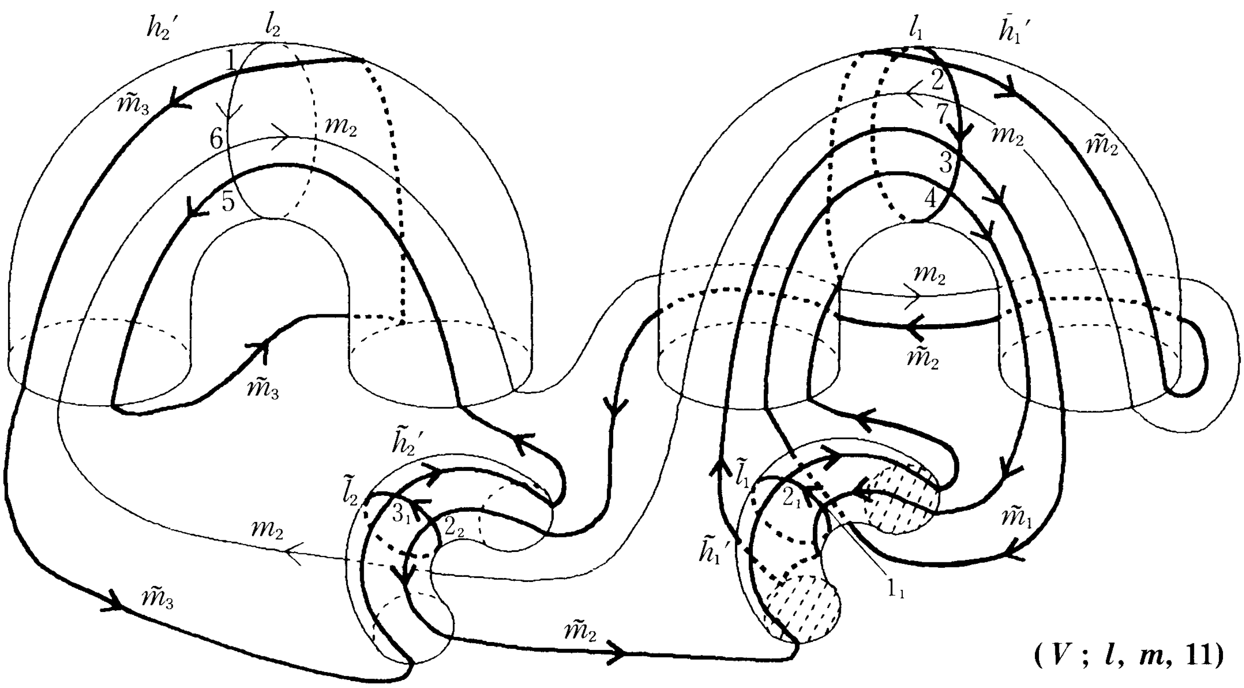
Fig. 2

Tfm $(U\,;\,m,\,l,\,7)$ (Transformation of $(U\,;\,m,\,l,\,7)$) ; cutting off the handle h_1 at the meridian disk $D_i(\partial D_i = m_i)$ and putting three handles $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$ over the two cutting places, $(U\,;\,m,\,l,\,11)$ of genus 4 is obtained.

Tfm $(V\,;\,l,\,m,\,7)$; adding two handles $\tilde{h}_1{}'$ and $\tilde{h}_2{}'$ to V, $(V\,;\,l,\,m,\,11)$ of genus 4 is obtained.



$(U\,;\,m,\,l,\,11)$

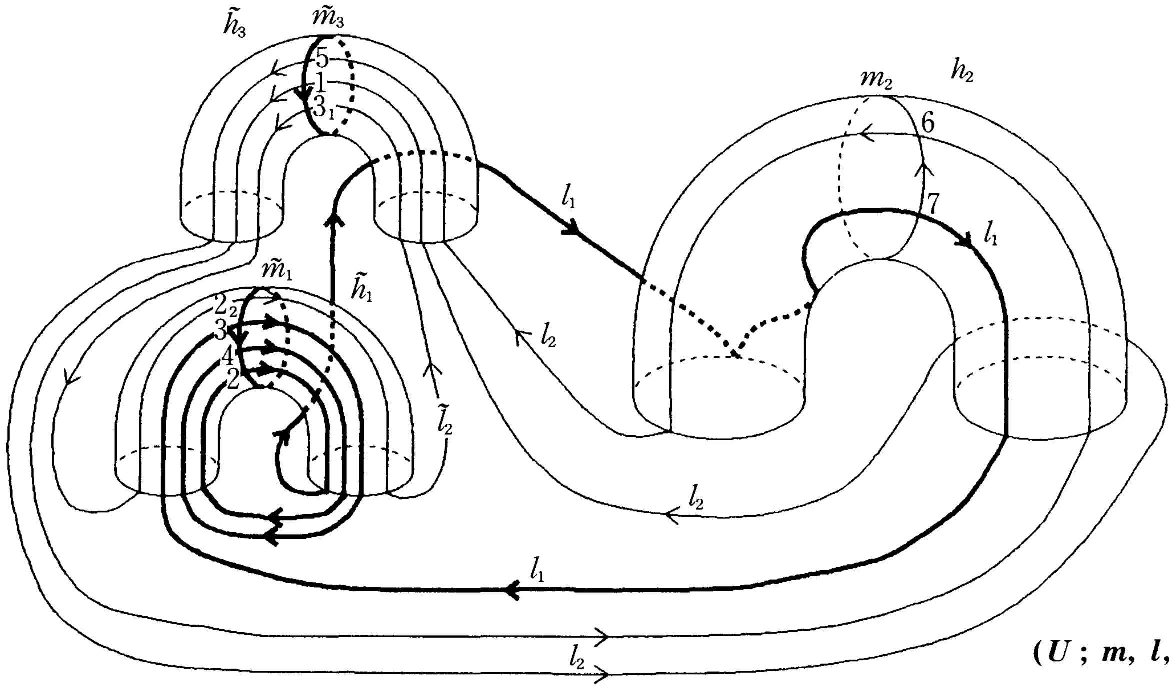


$(V\,;\,l,\,m,\,11)$

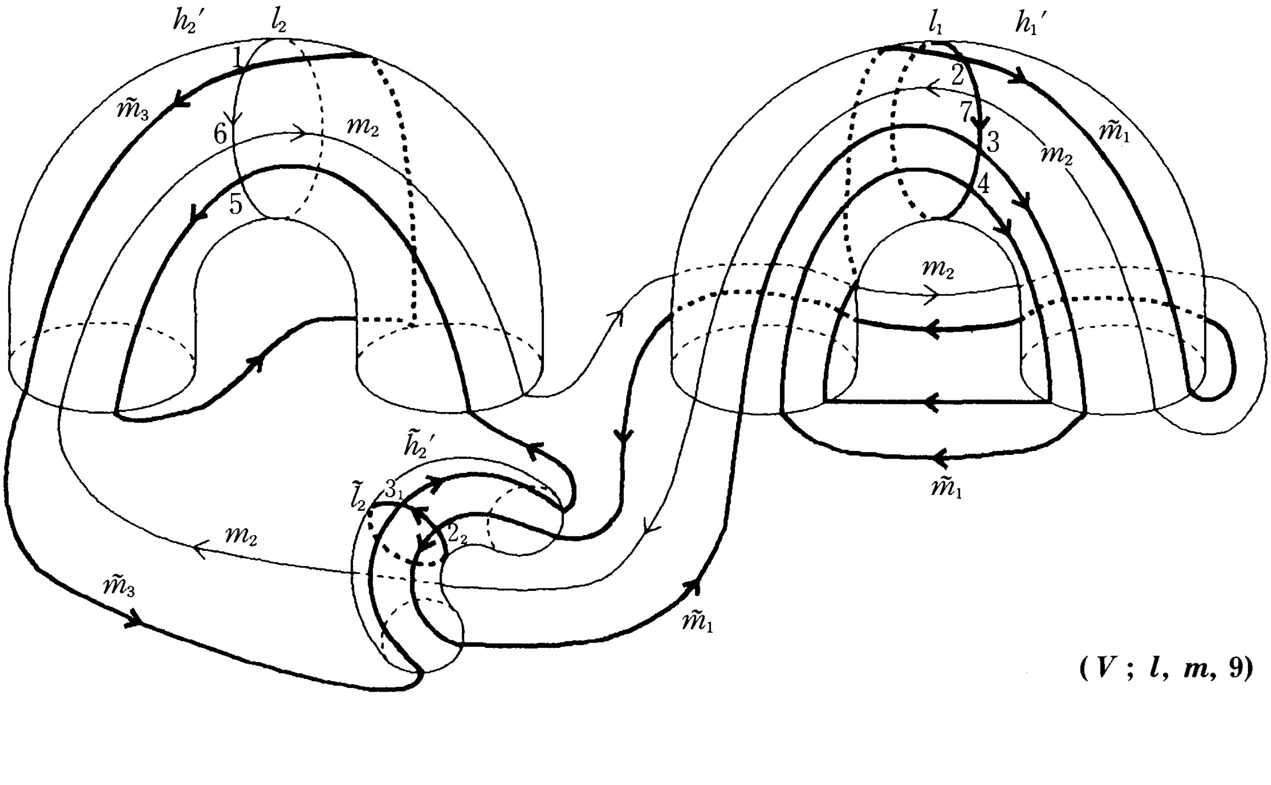
The H-cut-diagrams $G_2(m, l, 11) \cup G_2(l, m, 11)$ which are omitted here of $(U ; m, l, 11) \cup (V ; l, m, 11)$ have no-waves, respectively.

Tfm $(V ; l, m, 11)$; crushing the handle $\tilde{h}_1'$ at the meridian disk $\tilde{D}_1'$ ($\partial\tilde{D}_1' = \tilde{l}_1$) and cutting the deformed handle at the crushed disk (a pint), a genus 3 handlebody and a new circle is obtained. For the details of this way, refer to [**10**, p. 43]. Next putting the label $\tilde{m}_1$ on the circle and reorienting $\tilde{m}_1$, $(V ; l, m, 9)$ of genus 3 is obtained.

Tfm $(U ; m, l, 11)$; genus 3 H-diagram $(U ; m, l, 9)$ is obtained corresponding to the above transformation.



$(U ; m, l, 9)$



$(V ; l, m, 9)$

Dotted lines in $(U\,;\,m,\,l,\,11) \cup (V\,;\,l,\,m,\,11)$ show the method of transformations of the H-cut-diagrams corresponding to those of $(U\,;\,m,\,l,\,11) \cup (V\,;\,l,\,m,\,11)$.
The H-cut-diagrams $G_3(m,\,l,\,9) \cup G_3(l,\,m,\,9)$ of $(U\,;\,m,\,l,\,9) \cup (V\,;\,l,\,m,\,9)$ are Viro's case.

Example 5. Fig. 3 gives disconnected genus 1 H-diagrams $(U\,;\,m,\,l,\,0)$, $(V\,;\,l,\,m,\,0)$ of $S^2 \times S^1$ and those disconnected H-cut-diagrams $G_1(m,\,l,\,0)$, $G_1(l,\,m,\,0)$, respectively.

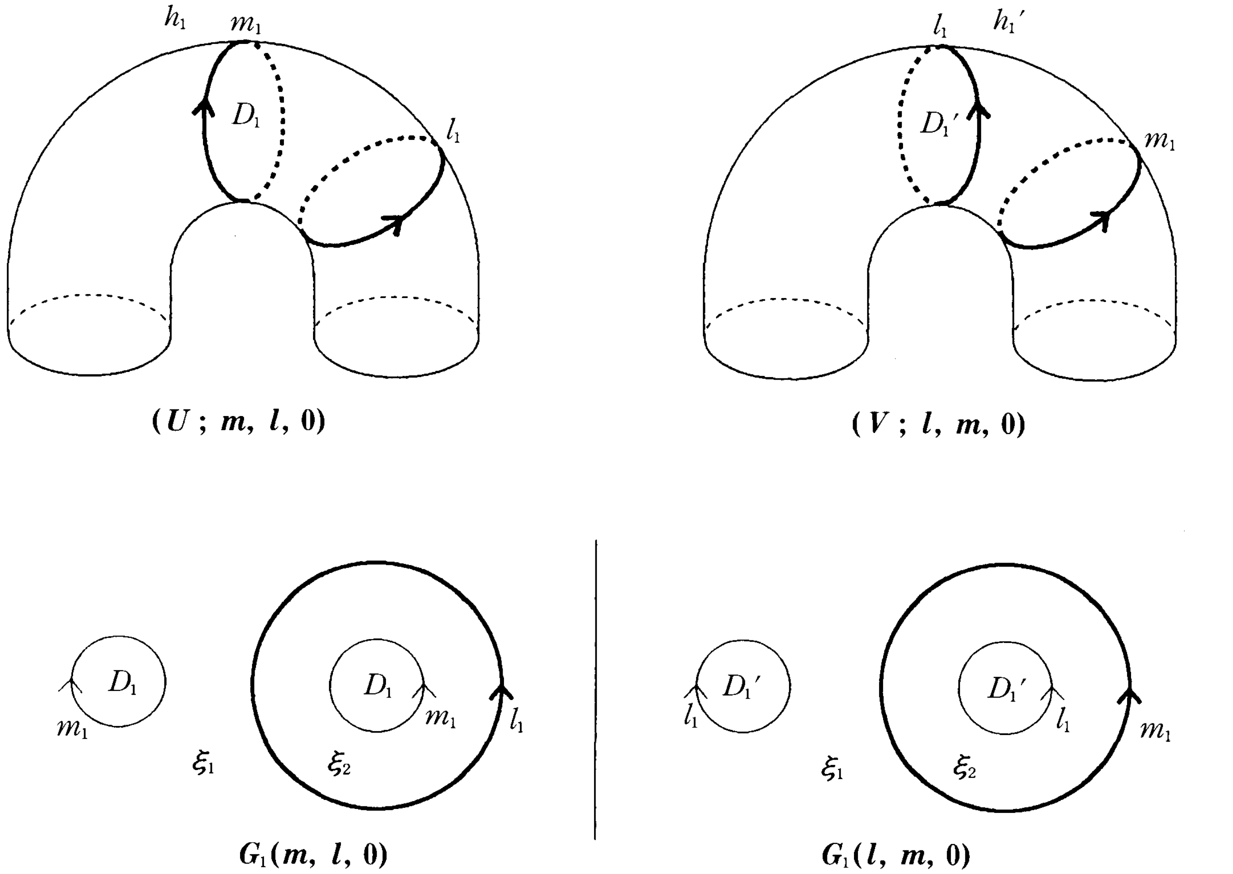
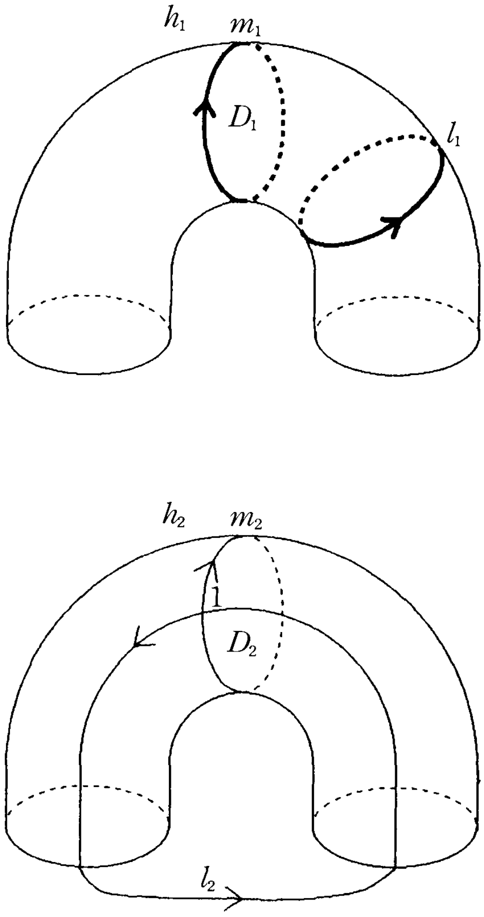


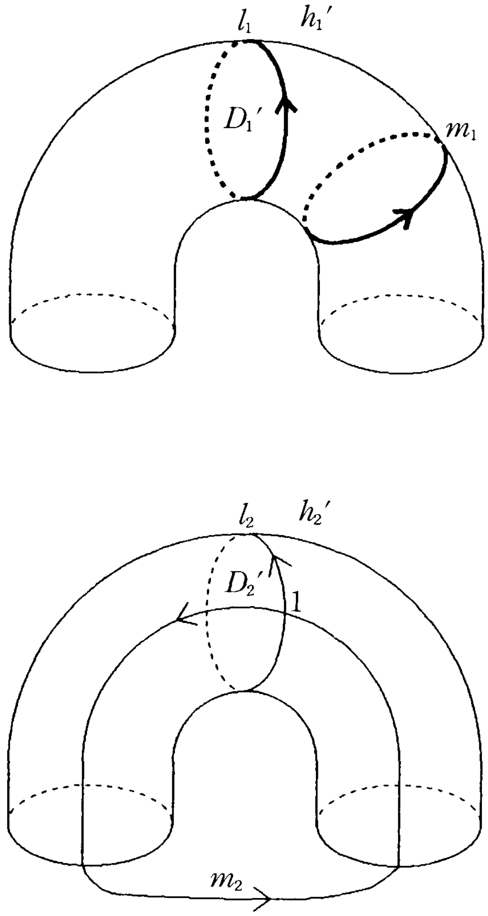
Fig. 3

We give transformations from the disconnected H-diagrams $(U\,;\,m,\,l,\,0) \cup (V\,;\,l,\,m,\,0)$ into connected ones.

Tfm $(U\,;\,m,\,l,\,0)$ and $(V\,;\,l,\,m,\,0)$; assume the canonical genus 1 H-diagram of S^3. If we take connected sum $(S^2 \times S^1) \# S^3$, then genus 2 disconnected H-diagrams $(U\,;\,m,\,l,\,1) \cup (V\,;\,l,\,m,\,1)$ of $S^2 \times S^1$ are obtained.



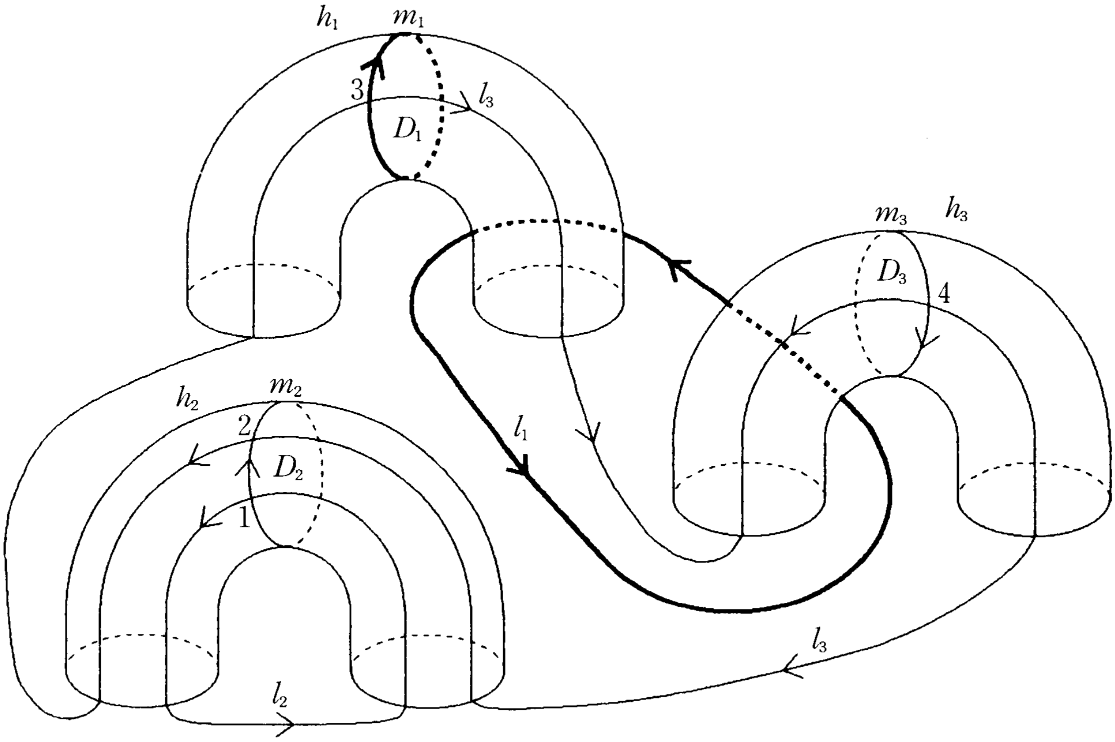
(U ; m, l, 1)



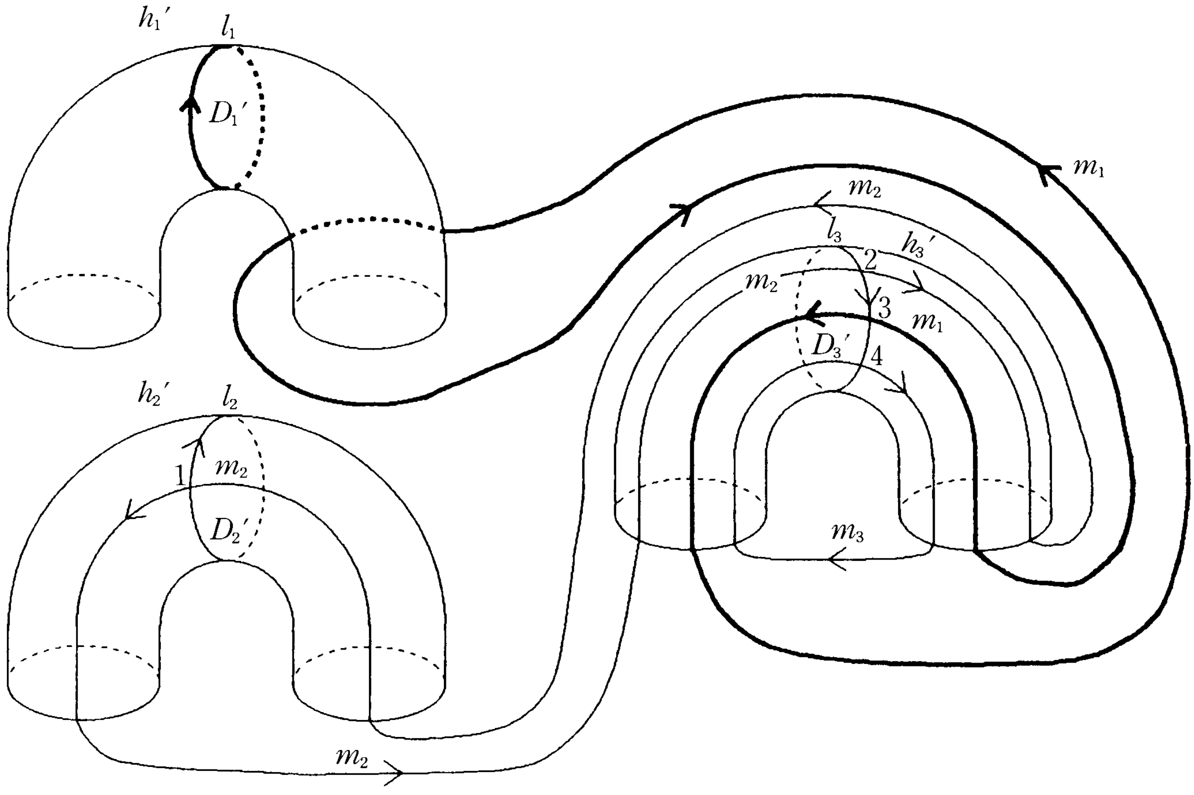
(V ; l, m, 1)

Tfm (U ; m, l, 1) ; add a handle h_3 to U so that a new longitude l_3 goes around the handles h_1, h_2 and h_3. Then genus 3 H-diagram (U ; m, l, 4) is obtained. Here, we draw the longitude l_1 once more on the spherical surface $\partial B_U{}^3$ because of the convenience.

Tfm (V ; l, m, 1) ; add a handle h_3' to V and draw the longitudes m_1, m_2, m_3 so that they become the dual diagram to (U ; m, l, 4). Then genus 3 H-diagram (V ; l, m, 4) is obtained.



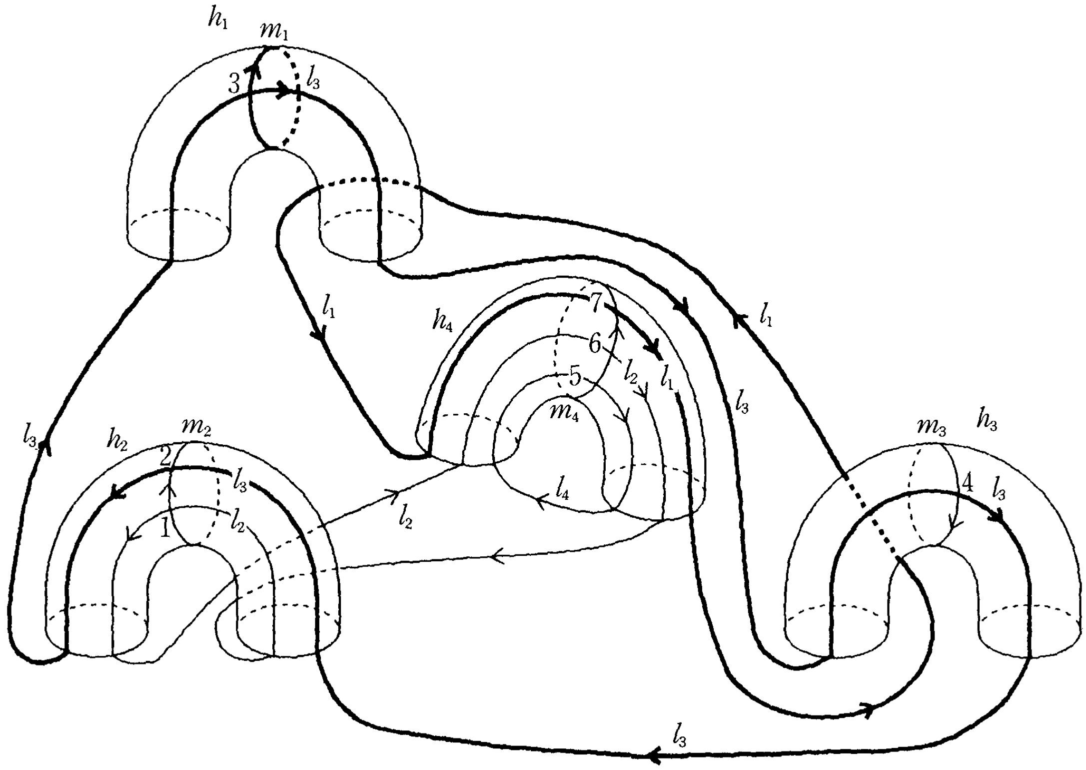
(U ; m, l, 4)



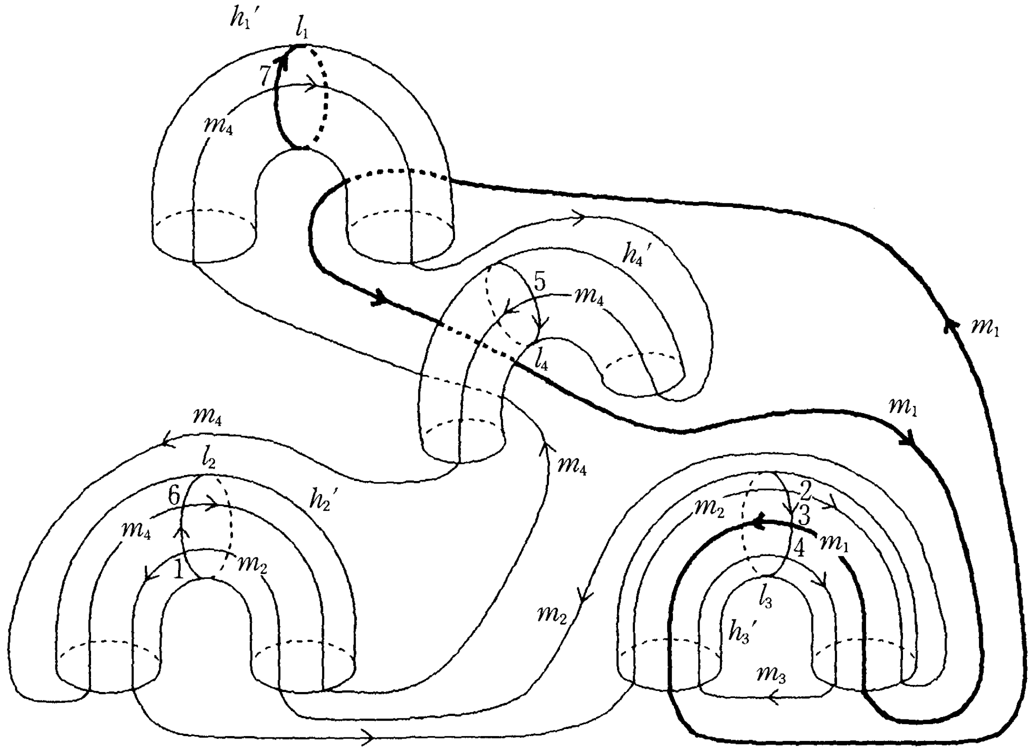
(V ; l, m, 4)

Tfm $(V\ ;\ l,\ m,\ 4)$; add a handle h_4' to V so that a new longitude m_4 goes around the handles h_1', h_2' and h_4'. Then genus 4 H-diagram $(V\ ;\ l,\ m,\ 7)$ is obtained.

Tfm $(U\ ;\ m,\ l,\ 4)$; add a handle h_4 to U and draw the longitudes l_1, l_2, l_3 and l_4 so that they become the dual diagram to $(V\,;\,l,\,m,\,7)$. Then genus 4 H-diagram $(U\,;\,m,\,l,\,7)$ is obtained. Fig. 1 is the H-cut-diagram of this.



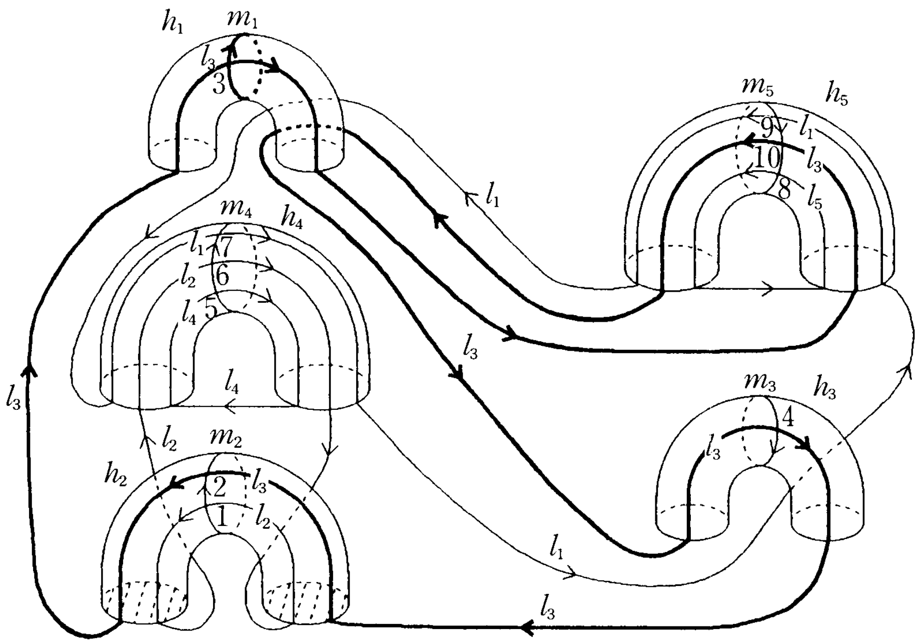
$(U\,;\,m,\,l,\,7)$



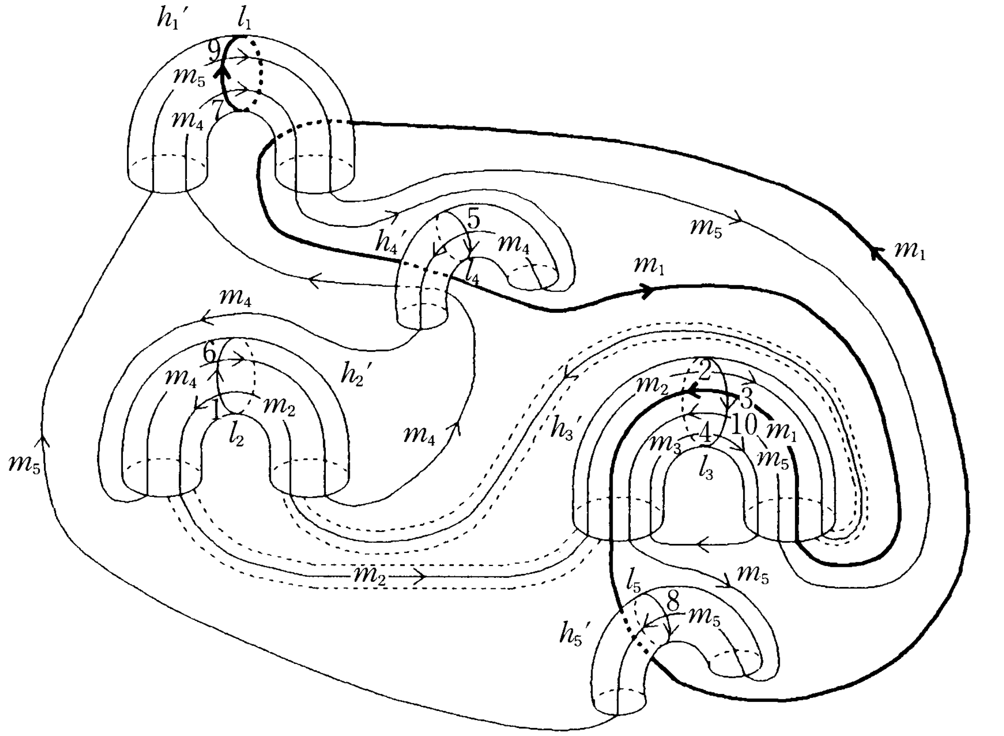
$(V\,;\,l,\,m,\,7)$

Tfm (V ; l, m, 7) ; add a handle h_5' to V so that a new longitude m_5 goes around the handles h_1', h_3' and h_5'. Then genus 5 H-diagram (V ; l, m, 10) is obtained. The H-cut-diagram of this is connected.

Tfm (U ; m, l, 7) ; add a handle h_5 to U and draw the longitudes l_1, l_2, l_3, l_4 and l_5 so that they become the dual diagram to (V ; l, m, 10). Then genus 5 H-diagram (U ; m, l, 10) is obtained.



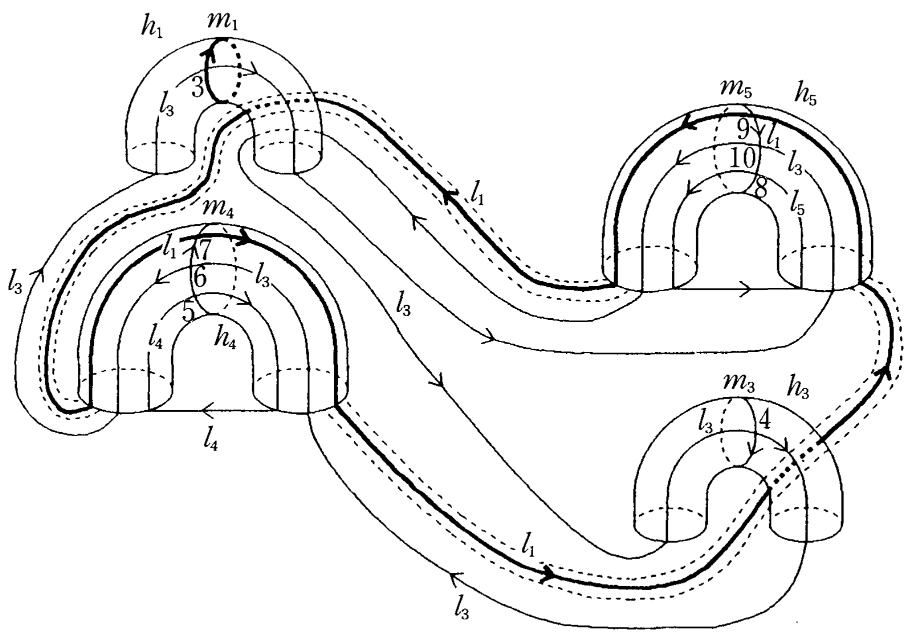
(U ; m, l, 10)



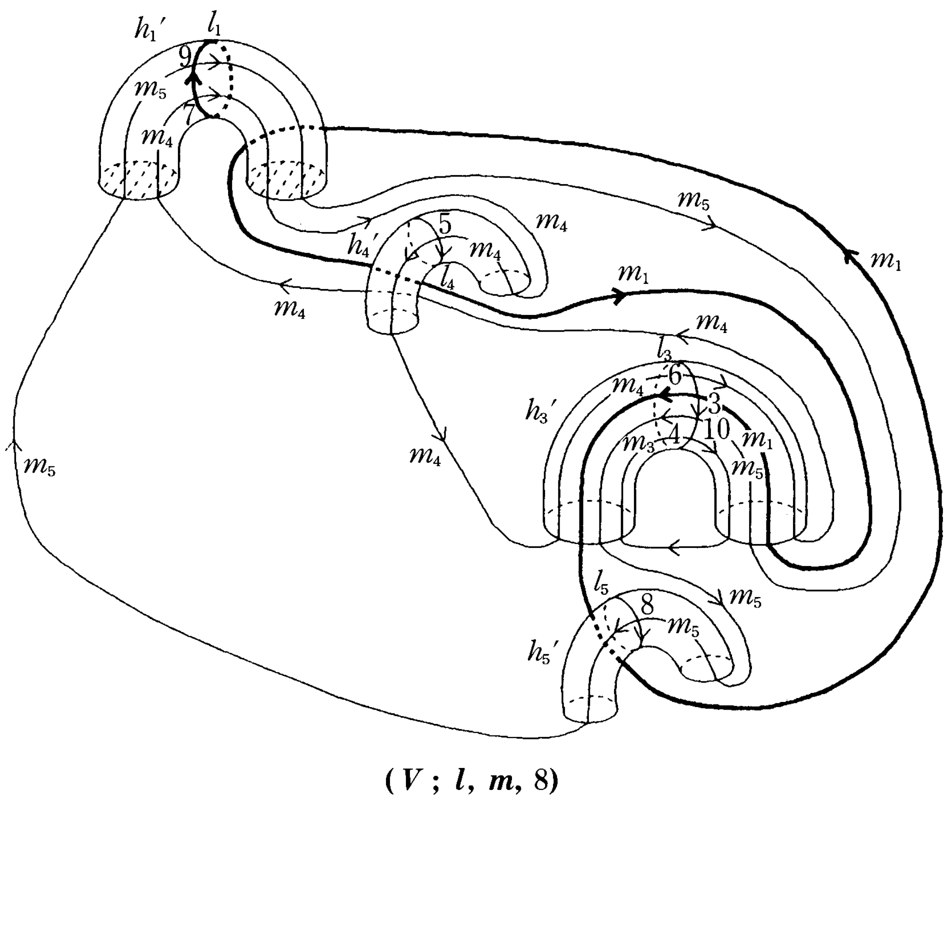
(V ; l, m, 10)

Tfm (U ; m, l, 10) ; crushing the handle h_2 at the meridian disk $D_2(\partial D_2 = m_2)$ and cutting the deformed handle at the crushed disk (a pint), a genus 4 handlebody and a new circle is obtained from l_2 and l_3. Next putting the label l_3 on it and reorienting it, genus 4 (U ; m, l, 8) is obtained.

Tfm (V ; l, m, 10) ; the dual genus 4 (V ; l, m, 8) is obtained corresponding to the above transformation.



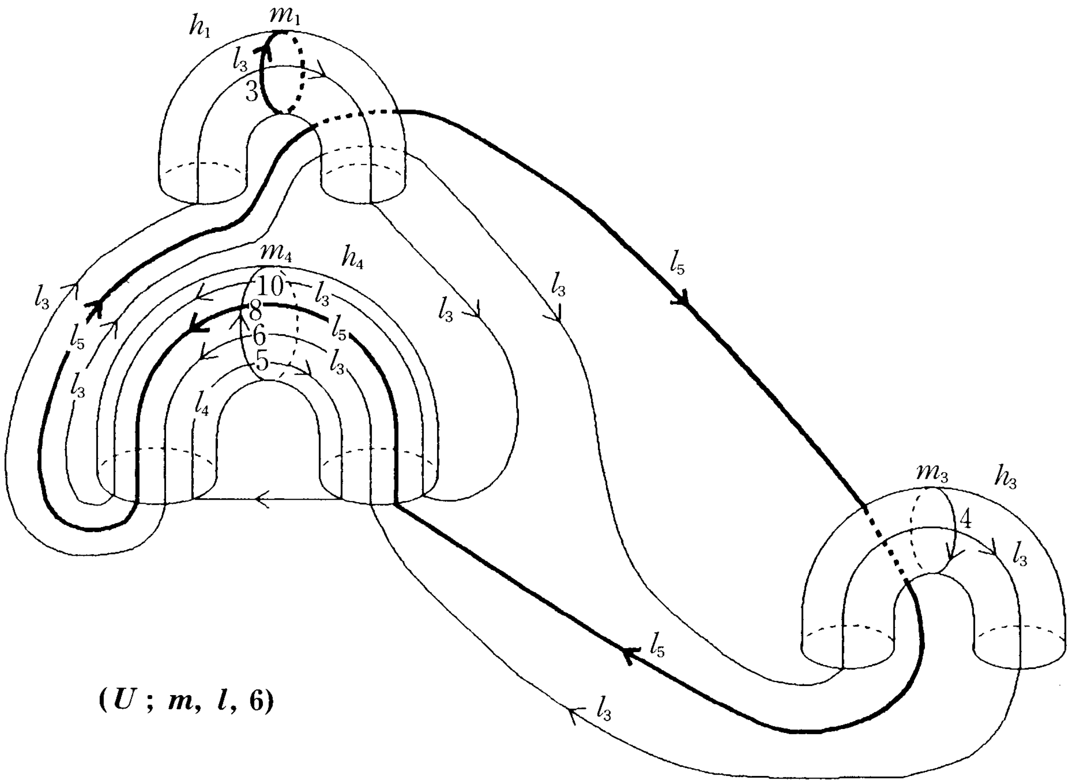
(U ; m, l, 8)



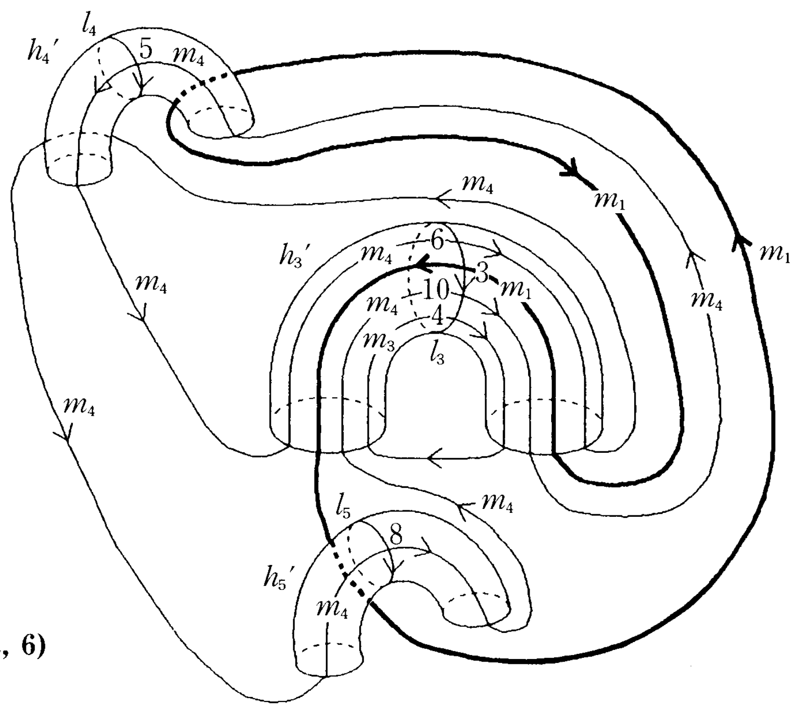
(V ; l, m, 8)

Tfm (V ; l, m, 8) ; crushing the handle h_1' at the meridian disk D_1' ($\partial D_1' = l_1$) and cutting the deformed handle at the crushed disk (a pint), a genus 3 handlebody and a new circle is obtained from m_4 and m_5. Next putting the label m_4 on it and reorienting it, genus 3 (V ; l, m, 6) is obtained.

Tfm (U ; m, l, 8) ; the dual genus 3 (U ; m, l, 6) is obtained corresponding to the above transformation. The H-(cut-)diagrams of these are connected.

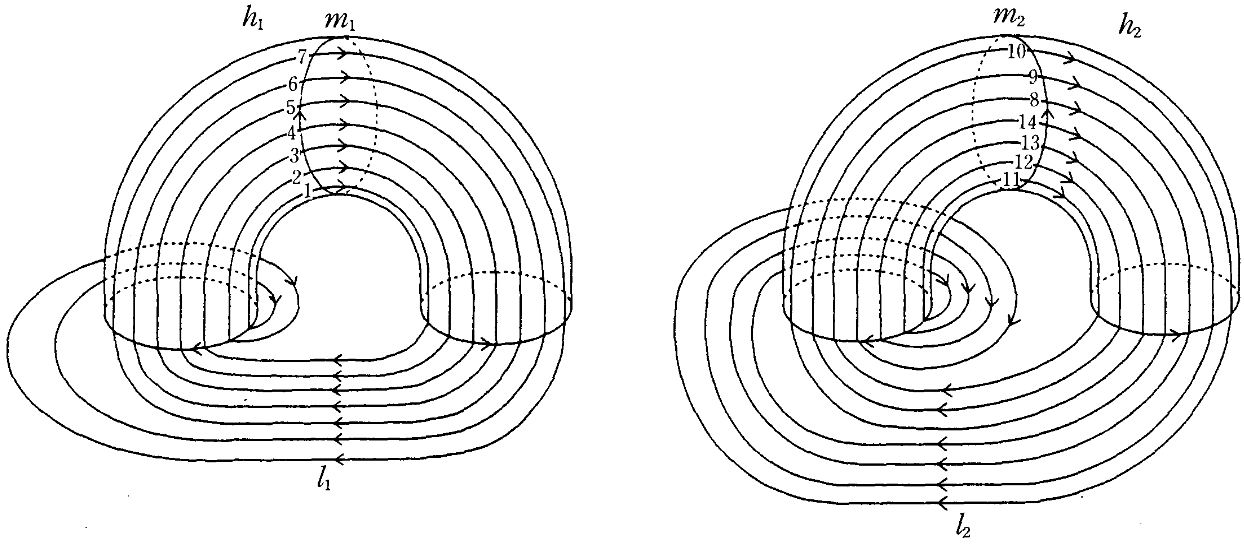


(U ; m, l, 6)

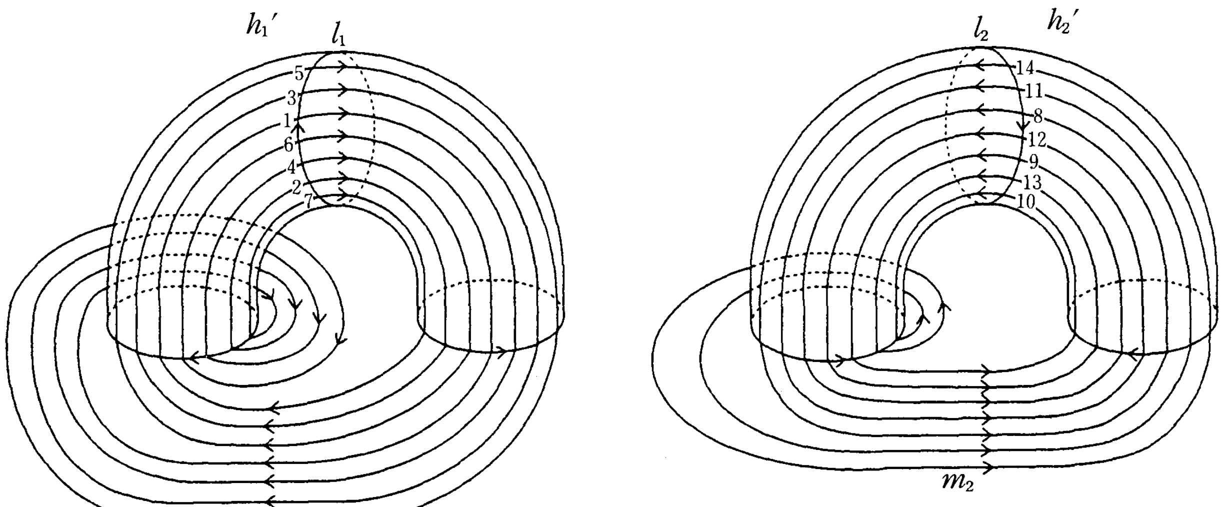


(V ; l, m, 6)

Example 6. Fig. 4 gives disconnected genus 2 H-diagrams $(U; m, l)$, $(V; l, m)$ of $L(7, 2) \# L(7, 4)$.



$(U; m, l, 14)$

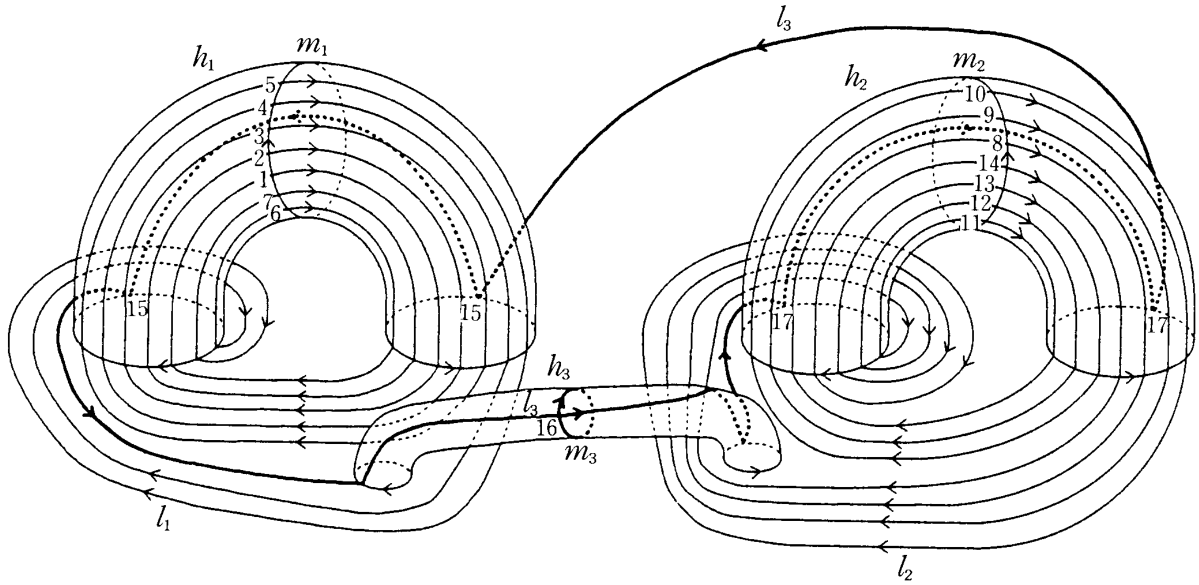


$(V; l, m, 14)$

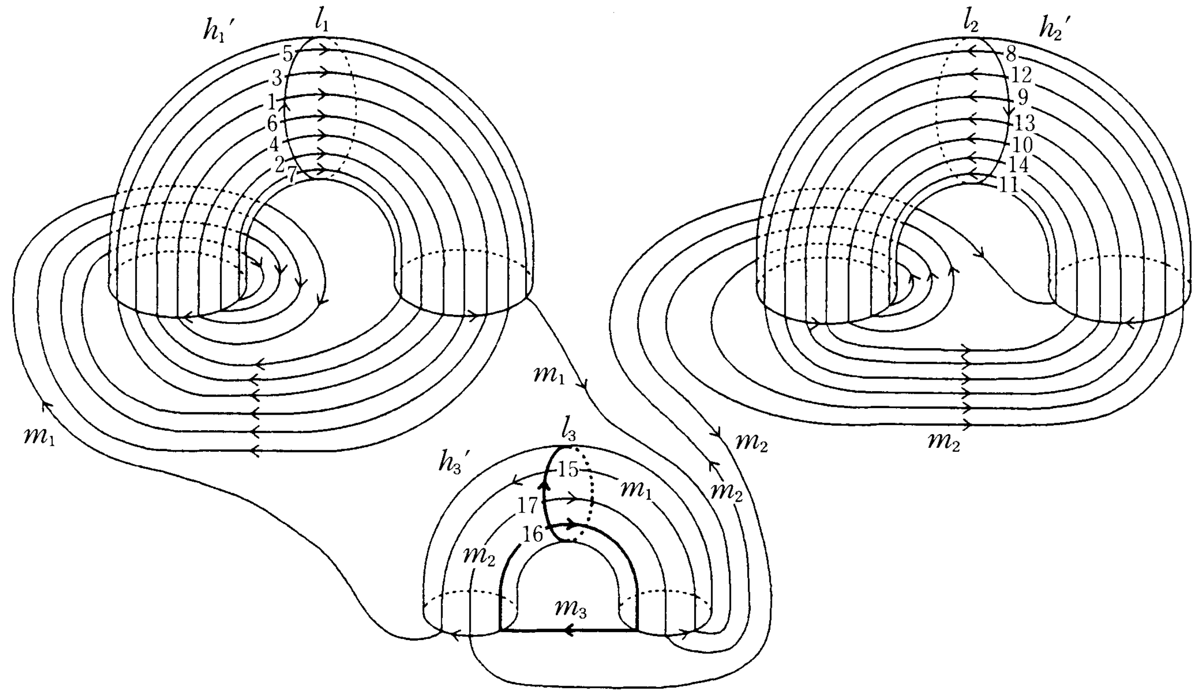
Fig. 4

Tfm (U ; m, l, 14) ; add a handle h_3 to U so that a new longitude l_3 goes around the handles h_1, h_2 and h_3. Then genus 3 H-diagram (U ; m, l, 17) is obtained. This H-(cut-)diagram is connected.

Tfm (V ; l, m, 14) ; the dual genus 3 (V ; l, m, 17) is obtained corresponding to the above transformation.



(U ; m, l, 17)



(V ; l, m, 17)

It is very difficult to transform H-diagrams as can be seen from the above examples. Therefore, it will be desirable to transform the H-cut-diagrams by using the *DS*-deformations instead of H-diagrams. Some examples in [8] will be helpful to master the transformations of H-cut-diagrams.

Transformations from $(1-A) \cup (1-A')$ into $(1-B1) \cup (1-B1')$ and corresponding handle sliding, longitude combining and changing in [**10**] are very important. Henceforth we will have to move forward those studies more.

References

[1] H. Ikeda & Y. Inoue: Invitation to DS-diagrams, Kobe J. Math., **2** (1985), 169-186.

[2] I. Ishii: Flows and spines, Tokyo J. Math., **9** (1986), 505-525.

[3] M. Yamashita: DS-diagram and Heegaard diagram (In Japanese), Publ. RIMS, Kyoto Univ., **636** (1985), 28-41.

[4] H. Ikeda, M. Yamashita, & K. Yokoyama: Deformations of DS-diagrams, In Topology and Computer Science: Proc. of the Symposium held in honor of S. Kinoshita, H. Noguchi and T. Homma, Edited by S. Suzuki, Kinokuniya Co. Ltd., Tokyo (1987), 81-138.

[5] S. Horiguchi: Transformations of Heegaard diagrams (In Japanese), Bulletin of Hakone Seminar (Low Dimensional PL Topology) (1990), 63-109.

[6] ___________: Transformations of Heegaard diagrams on nonequivalent Heegaard splittings L(7, 2) # L(7, 2) and L(7, 2) # L(7, 4), Journal of Aomori Univ. & Aomori Jr. Coll., **14** (1991), 57-85.

[7] ___________: Transformations of the fundamental group associated with Heegaard diagrams, Bulletin of Niigata Sangyo Univ., **9** (1993), 52-107.

[8] ___________: Transformations of the fundamental groups associated with Heegaard diagrams IV, Bulletin of Niigata Sangyo Univ., **13** (1995), 81-133.

[9] ___________: Transformations of the fundamental groups associated with Heegaard diagrams V, Bulletin of Niigata Sangyo Univ., **14** (1995), 55-95.

[10] ___________: Transformations of Heegaard diagrams corresponding to those of Heegaard cut diagrams, Bulletin of Niigata Sangyo Univ., **15** (1996), 29-86.

[11] J.H.C. Whitehead: On certain sets of elements in a free group, Proc. London Math. Soc. (2), **41** (1936), 48-56.

[12] I.A. Volodin, V.E. Kuznetsov, & A.T. Fomenko: The problem of discriminating algorithmically the standard three-dimensional sphere, Russian Math. Surveys, 29: 5 (1974), 71-172.

[13] O. Ja. Viro: The Volodin-Kuznetsov-Fomenko conjecture on Heegaard diagrams is false, Uspeckhi Mat. Nauk, **32** (1977), 175-176.

[14] J.S. Birman: Heegaard splittings, diagrams and sewings for closed, orientable 3-manifolds, Lecture notes for CBMS conference at Blacksburg, Va., Oct. 8-12, 1977.

[15] H. Terasaka: On circles in closed surfaces (In Japanese), Publ. RIMS, Kyoto Univ., **219** (1974), 70-89.

[16] T. Homma: On Heegaard splittings and circles in surfaces (In Japanese), Publ. RIMS,

Kyoto Univ., **219** (1974), 90-102.
[17] T. Homma, M. Ochiai, & M. Takahashi : An algorithm for recognizing S3 in 3-manifolds with Heegaard splittings of genus two, Osaka J. Math., **17** (1980), 625-648.
[18] O. Morikawa : A counterexample to a conjecture of Whitehead, Math. Seminar Notes of Kobe Univ., 8 (1980), 295-298.
[19] M. Ochiai : Construction of 3-manifolds by computer and its treatments by graphics (In Japanese), Mathematical Sciences, **256**, Saiensu-Sha Co. Ltd., Tokyo (1984), 22-29.
[20] J. Hempel : 3-manifolds, Ann. of Math. Studies 86, Princeton Univ. Press. 1976.
[21] H. Seifert & W. Threlfall : A Textbook of Topology, Translated in English by M.A. Goldman, Academic Press, Inc. 1980.

Shunzi HORIGUCHI
Niigata Sangyo University
Niigata
Japan